THE EISENHOWER LIBRARY
3 1151 02717 1002

150,776

Library

of the Johns Hopkins University

Orthocentric Properties of the Plane Directed N-Line.

Dissertation

Submitted to the Board of University Studies of the Johns Hopkins University in Conformity with the Requirements for the Degree of Doctor of Philosophy.

by

Joseph Ellis Hodgson

150,776

The writer desires to express his gratitude for the interest, encouragement, and assistance given him by Prof. Morley in the preparation of this paper and throughout his entire course at the University.

Introduction.

In a memoir* entitled "Orthocentric Properties of the Plane n-line" Prof. Morley has generalized for n lines of a plane metrical analogues to the elementary theorems that the perpendiculars of a triangle meet in a point, and that the four such points, or orthocentres, for the four triangles of a 4-line lie on a circle.

The first of a chain of theorems proved in the Memoir just mentioned is, that if from the vertices of any 3 of the lines perpendiculars be let fall upon the remaining sides, the 4 perpendiculars meet at a point. It was further shown that for an odd num-

* Transactions of the American Math. Society, Vol. 4, No. 1, pp 1-12.

ber of lines there is an obstacle, while for an even number there is a zone of stability, or, as it may be termed, a valley.

In his lectures on Vector Analysis during the session of 1907-8 to the students of the Mathematical Department of the Johns Hopkins University, Prof. Morley proved that if a perpendicular be let fall from the center of the inscribed circle of any triangle upon the lines being three of these curves*, upon the lines left on each way, the four perpendiculars to each a circle.

Now, there is such correspondence, in a way, to the points and circles each perpendicular, the locus of the remaining circle, from the three each center of a way 3 of the lines,

* Oeuvres de Laguerre, Tome II., p. 592, Sur la Géométrie de direction.

when the limits are part of specified
constants.

The class of theorems proved in
this memoir above I propose to call
in connection with the theorems
treated in my first paper, Report
the class of theorems formed by
their ∞.

As a matter of interest these
theorems will be established from
at least two points of view, namely
in connection with linear forms whose
coefficients are circles involving in them-
selves the deviation of one or more of
the circles will be considered, in or-
der to point out some of the relations
existing among the various kinds
of the hyperbolic. Moreover, determinants
of circle coordinates will be introduced

and there must be taken up some discussion of some details,

In the derivation of a number of interesting results [illegible] * using [illegible] as follows, [illegible] we [illegible] the [illegible] of [illegible] the following form.

$$x + \frac{1}{x} = \text{[illegible]}$$

[illegible] x [illegible] $\bar{x}$ [illegible] the [illegible] circular or conjugate coordinates [illegible] by the equations

$$x = X + iY,$$
$$\bar{x} = X - iY,$$

X and Y being the rectangular coordinates [illegible], a quantity, and t a quantity, or term, of absolute value 1, t, therefore, moves on the unit circle. In this manner a [illegible] correspondence [illegible] between [illegible] and [illegible] is established [illegible]

* Trans. American Math. Society, Vol. 1, No. 3, pp. 323-338

the line has its equation in the above form, the slope, or climant, of the line is given by $-\frac{1}{t}$. This is certainly a condition, a necessary one in such, that a direction may be given to a line to the exclusion of its opposite direction, and it is also sufficient when the line is assumed in direct and not retrograde to some curve. When, however, the line refers to a curve by its equation, the further condition that the climant of the point to the point of contact of the line with the curve shall be the direction whether $+t$ or $-t$ be used as parameter seems to be required.

The locus of the centers of all circles tangent to two given circles when these have real homogeneous direction is the locus of the centers of

other point of intersection. If, however, we give direction to the three axes so that the tangent circle of each has at the point of contact the same direction as the line itself, only one of the circles will be the locus of points equidistant from the two directed lines.

Next, consider three directed lines, as in the three corresponding equidistant points. These meet in a point, which is the only point equidistant from the three lines. This point may, or may not, be the inscenter of the 3-line; if it is not, it is one of the ex-centers. The point is that however the lines may be given in any of the eight directions, there is only one point e-

equidistant from the 3 given lines.

It is often advantageous in any investigation concerning the metric properties of lines in a plane to take the lines as those of curves suitable for the purpose of the investigation at hand. In general, the curves so employed must of course be of such class as to admit of their lines, considered as lines in a plane merely, being used without loss of generality.

Part I.

When the Lines are Tangents of an A^{2n}.

Section I.

The Orthocentric Circle for $2n$ Lines.

The Directrix for $2n+1$ Lines.

For the purpose of showing that the four perpendiculars cut off from the equivalent points of every 3 of 4 lines of specified direction upon the lines left out in each case are on a circle the line equations in conjugate coordinates of the asteroid, or hypocycloid of Class 4,

$$t^4 - x t^3 - \bar{x}\, t + 1 \quad ,$$

may be combined with that of the general circle in the same coordinates

$$t^2 (x - \bar{x}) + (\bar{y} - \bar{z}) - z = 0 \, ,$$

where z is real. It is to be noted that the element of each of these lines is $-\frac{1}{t^2}$, and hence adapted to the case of lines of direction. Moreover this combination of the line equations detects from that 4 common

lines of the two regions, a set of 7, and with each of these may be connected a direction, e.g. the tangent at t_i is $-\frac{1}{t_i^2}$, $i = 1, \ldots, 7$. Each of the roots of the resulting equation on the elimination of x and y from the equations of the two curves. In this case the element $-\frac{1}{t_i^2}$ belongs to the lines of either the outside or inside, when the substitution is made for t_i.

However, instead of the method, the more formal one parallel to it as a curve, known as the parametric, may be given. The equation of this curve is

$$t^7 - xt^5 + yt^2 - \bar{x}\bar{t} + \bar{y} = 0,$$

where in its real form this equation is to be self-conjugate. If so

convenient to refer to this curve as the evolute of the curve A^4. That the A^4 is, in fact, the evolute is seen from the fact that the distance from the point t on the curve to the point $\frac{t^4+4t^2+1}{t^3+t}$ of the curve is

$$\frac{t^4+4t^2+1}{t^3+t} - \left(t + \frac{1}{t}\right) = \frac{3}{t^3+t},$$

The A^4 is, therefore, parallel to the curve for which $c = 0$, that is, the curve $t + \frac{1}{t}$. Similarly, the curve

$$t^6 - 5t^4 + 8t^2 - 4 = (t^2+1)(t^2-2)^2$$

of the same general type as the parabola may be written as A^6. The 5's in the expression are symmetric functions of the t's and so can be expressed in terms of the coefficients. Also, the A^6 is parallel to the curve

$$t^6 - 5t^4 + 7t^2 - 2 = (t^2+1)(t^2-1)^2.$$

Let us now consider the equation of the parabola and general circle

(1) $t^4 - \lambda t^3 + \mu t^2 - \bar{\lambda} t + 1 = 0$

(2) $t^2(\lambda - a) + (\bar{\lambda} - \bar{a}) = 2\rho t$.

Multiply the latter by t and combine. There results

(3) $t^4 - at^3 + (\mu - 2\rho) t^2 - \bar{a} t + 1 = 0$

which gives 4 common directum lines of the two curves, corresponding to the 4 roots of (3).

In (3) $a = S_1$, $(\mu - 2\rho) = S_2$, $\bar{a} = S_3$, $1 = S_4$, the S's being symmetric functions of the roots of (3). It is to be noted that $S_4 = 1$ always, whatever the two curves (1) and (2) may be. This is accounted for by considering the equations of the curves to be homogeneous.

Since, in (2), a is the center of the circle, the general circle of the plane, and since, in (3), $a = S_1$, the center of this circle of 3 of 4 lines of the

set of lines is

(4) $\quad x = t_1 + t_2 + t_3 + t_4 = S_1,$

but since $S_4 = t_1 t_2 t_3 t_4 = 1$, $t_4 = \frac{1}{t_1 t_2 t_3}$.

Hence, the center of the circle is given by

$$x = t_1 + t_2 + t_3 + \frac{1}{t_1 t_2 t_3}$$

(a) $\quad$ or $\quad x = S_1 + \frac{1}{S_3}$

where, of course, the S's refer to three t's; and since the circle touches all 4 lines there are 3 other expressions similar to (a), each giving the center of the circle. Symmetrizing (a) for the fourth t, we have

(5) $\quad x = S_1 - t + \frac{t}{S_4}$,

(5) is the equation of a circle whose center is S_1 and whose radius is $\left|1 - \frac{1}{S_4}\right|$. It is the circle in the 4 quadrilateral formed of the 4 triangles of 3 of 4 lines of general direction. Now, every fourth line of the direction

locus of the 4th point given by the parameter t_4, say, is

$$t_4^4 - s_1 t_4^3 + s_2 t_4^2 - s_3 t_4 + 1 = 0,$$

while a perpendicular to this line, at t_4 is of the form

$$x t_4^2 - \bar{x} = \lambda ;$$

and if this locus is to contain the inpedantant point of the 3 points given by the parameters t_1, t_2, t_3, that is

$$s_1 + \frac{1}{s_3}$$

we have

$$x t_4^2 - \bar{x} = t_4^{-1}\left(s_1 + \frac{1}{s_3}\right) - \left(\frac{s_2}{s_3} + s_4\right), \tag{6}$$

since the symmetric s_1 is $\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3}$

$$= \frac{t_2 t_3 + t_3 t_1 + t_1 t_2}{t_1 t_2 t_3} = \frac{s_2}{s_3}.$$

Equation (6) can be transformed into the equation (dropping suffix 4 of t)

$$x t^2 - \bar{x} = t^{-1}\left(s_1 - t + \frac{t}{s_4}\right) - \left[\frac{(s_2 - s_1 t + t^2)t}{s_4}\right] + s_3 - t + t_1 t_2 t.$$

By symmetrizing the symmetric function s_1, s_2, s_3 for 3 t's, we mark if this sym-

methodizing is required in the course of work, we may write out that for S_1, S_2, S_3 for 3 t's, so that no detailed steps need hereafter be given.

S_1 for 3 t's is symmetrical in all that for some t_4 is written

$$t_1 + t_2 + t_3 + t_4 - t = S_1 - t,$$

so that when the variable t takes on successively the values t_1, t_2, t_3, t_4 the preceding just written drops back to S_1 for t_2, t_3, t_4; t_1, t_3, t_4; t_1, t_2, t_4; t_1, t_2, t_3 respectively. Also, S_2 for t_1, t_2, t_3 is

$$t_1 t_2 + t_1 t_3 + t_2 t_3.$$

For this write

$$t_1 t_2 + t_1 t_3 + t_1 t_4 + t_2 t_3 + t_2 t_4 + t_3 t_4 - (t_1 + t_2 + t_3 + t_4)\, t + t^2.$$

When now t takes the values t_1, t_2, t_3, t_4 this symmetrical expression reduces to S_2 for every set of 3 t's that can be chosen from the four. Most succinctly for

the symmetrizing of any one expression in symmetric functions of the t's. S_2 for 3 t's is naturally symmetrized for 4 t's by writing for S_2 the expression $\frac{2S_4}{t}$.

Returning now to equation (7), where t takes in succession the values t_1, t_2, t_3, t_4, (7) reduces to the equation of each of the 4 permutations, i.e. that giving (6), in case for each of these 4 sets of 3 t's selected from $t_1, t_2, t_3 t_4$. In other words, we have in (7) the equations of the 4 pairs of lines from the 4 equivalent foci of the tri-angles of 3 of 4 points when more left out in each case. Dis-carding (7)' we have at once

(7) $\quad (\lambda - \frac{S_1}{S_4})t^2 - (\bar{\lambda} - S_3) = t(\mu - \frac{S_2}{S_4})$,

the line equation in conjugate coordinates of a circle. Hence we have the result of a theorem of Steiner that if

from each of the 4 equidistant p[illegible] of 3 of 4 directed lines, a perpendicular be let fall upon the line left out in each case, the four perpendiculars are on a circle.

Since this is a circle on perpendicular lines, we may for convenience of reference call it the orthocentric circle of the directed 4 lines.

The center of this circle is $\frac{S_1'}{S_4}$; its radius is $\frac{1}{2}\left|S_2\left(1-\frac{1}{S_4}\right)\right|$. But the absolute value of S_1 is equal to that of $\frac{S_1}{S_4}$, since each is $\sqrt{\frac{S_4}{S_4}}$ also. So, S_1 was the center found for the circle on the 4 equidistant points of 3 of 4 lines, while the origin is the center of the A^4 as given by equation (1). We conclude, therefore, that the center of the A^4, of which 4 directed lines are 4 more, is equidistant from the center of the circle on the 4 equidistant

points of the two triangles of these lines and the center of the circle touched by the perpendiculars to the 4th line from each of the equidistant points. Or, calling the circle on the equidistant points the Feuerbach circle of the 4 lines, we may state the preceding theorem this way: The center of the A^4, of which 4 lines are given, is equidistant from the centers of the Feuerbach and orthocentric circles of the 4-line.

The point, or map, equation of

$$\left(x - \frac{S_1}{S_4}\right)t^2 - (\bar{x} - S_3) = t\left(S_2 - \frac{S_2}{S_4}\right)$$

is

$$2T\left(x - \frac{S_1}{S_4}\right) = \left(S_2 - \frac{S_2}{S_4}\right)$$

in which the parameter is denoted by T. Symmetrizing this equation we have

$$(10)\quad 2T\left(x - \frac{(S_1 - t)t}{S_5}\right) = \left(S_2 - S_1 t + t^2\right)\left(1 - \frac{t}{S_5}\right)$$

where the S's are for 5 t's. But from

the relation for t becomes

$$S_2 t - S_1 t^2 + t^3 = S_3 - \frac{S_4}{t} + \frac{S_5}{t^2}.$$

Hence (10) may be written as

(11) $$X = \frac{(S_1 - t)\,t}{S_3} + \frac{1}{2T}\left[(S_2 - S_1 t + t^2) - \frac{1}{S_3}\left(S_3 - \frac{S_4}{t} + \frac{S_5}{t^2}\right)\right]$$

If now t takes the values t_1, t_2, t_3, t_4, t_5, (11) gives the X-abscissæ of the five S-points, one for each of the t values substituted on the right. Moreover, if t be replaced by $t - \frac{1}{t^2}$ and terms of the 4th and higher orders be neglected, (11) becomes an expression with the parameters, t and T, and it contains the S-orthocentric circles. The question arises as to what are the loci of these circles. For the purpose of ascertaining the answer to this question differentiate (11) with t and multiply the result by t. This done gives

$$t\,\frac{\partial}{\partial t} X = \frac{(S_1 - 2t)\,t}{S_3} + \frac{t}{2T}\left[(-S_1 + 2t)\,t - \frac{1}{S_3}\left(\frac{S_4}{t} - \frac{2S_5}{t^2}\right)\right].$$

Again differentiate the expression as to T and multiply the result by T.

$$T D_T^2 K = -\frac{1}{2T}\left[(S_2 - S_1 t + t^2) - \frac{1}{S_5}\left(S_3 - \frac{S_4}{t} + \frac{S_5^2}{t^2}\right)\right]$$

The complement of $t D_T^2 K$ is

$$\left(\frac{4}{T} - \frac{2S_5}{t^2}\right) + \frac{1}{2T}\left[\left(-\frac{2a}{S_5 t} + \frac{2}{t^2}\right) - (S_1 - 2t)\right]$$

that for $T D_T^2 K$ is

$$-\frac{1}{2T}\left[\left(\frac{2}{S_5} - \frac{4}{t} + \frac{1}{t^2}\right) - (S_1 - 2t)\right].$$

By subtracting the latter from

$$t D_t^2 K - T D_T^2 K = \text{the complement},$$

which is the equation of condition for a central conic referred to conjugate diameters as axes, we obtain

$$\frac{(S_1 - 2t)tT}{S_5} + \left[\frac{1}{2}(S_1 t + t^2) - \left(\frac{S_4}{S_5 t} - \frac{2}{t^2}\right)\right]$$

$$+ \frac{1}{tT}\left(S_4 - \frac{2S_5^2}{t}\right) + \frac{1}{2}\left[\left(-\frac{S_4}{S_5 t} + \frac{2}{t^2}\right) - (S_1 t - 2t^2)\right] =$$

or, rearranging the terms,

$$\left(\frac{S_1 - 2t}{S_5}\right)tT + \frac{1}{tT}\left(S_4 - \frac{S_5^2}{t}\right) - \frac{1}{2}(S_1 t - 2t^2) = \frac{1}{2}\left(\frac{2}{S_5} - \frac{2}{t}\right) = 0$$

(12) is a quadratic in T, of which one root is evidently $T = S_{5-}$. Hence we can write (11) $\frac{1}{2}t - t^2$ for shortness as $\frac{2t-2S_{5-}}{t^2}$ m. Then (12) is $\frac{1}{S_{5-}} m T^2 - (m + \frac{1}{S_{5-}} m) T = -m$ from which

$$T = S_{5-} \text{ and } \frac{m}{m}$$

Restoring the values of m and m, we have as the 2nd root of the quadratic

$$\frac{2t}{t^2}(S_1 - \tfrac{1}{2}t)$$

Both roots are $\frac{1}{t}$ when $t = 0$. If we take these values for k in (11), the resulting equation in t will give the intercepts of the tangents corresponding to these roots substituted. Considering first the intercepts when k has the value $T = S_{5-}$ in (11), we obtain

(13) $k = \frac{(\lambda - t_2)t_2}{S_{5-}} + \frac{1}{2S_{5-}}(\lambda - \tfrac{1}{2}t + t^2) - \frac{1}{2}(\lambda - \frac{\lambda}{t} + \frac{\lambda}{t^2})$, or

(14) $k = \frac{2\lambda - t}{2}\frac{t}{S_{5-}} - \frac{1}{2S_{5-}}(\lambda - \frac{\lambda}{t} + \frac{\lambda}{t^2})$

Let us obtain the equation of the envelope by eliminating t from (14) and its derivative. Writing for shortness $\frac{1}{2S_{5-}} = a$,

(14) [illegible]

(15) $\bar{x} = \frac{1}{2}\left(s_1 + \frac{s_4}{t} - \frac{s_3}{t^2}\right) - \frac{1}{2}\,\frac{1}{s_2^2}\left(\frac{s_3^2}{s_2^2} - \frac{2\,s_3 t}{s_2^2} + \frac{t^2}{s_2^2}\right)$

Combining these equations, we have

(16) $$x - \frac{1}{s_2^2}\,\bar{x} = \frac{s_3^2}{2\,s_2^2} - \frac{s_3}{s_2^2}\,t,$$

the equation of a line in the xt-plane. Hence we see that the [illegible] the [illegible] of the [illegible] are on a line.

That part of the envelope of the family of [illegible] corresponding to the [illegible] is not [illegible].

The extraneous circle for from the value $t = \infty$ is

$$\left(x - \frac{s_3}{2s_2}\right)^2 + \left(\bar{x} - s_1\right)^2 = \left(\frac{s_3}{2s_2}\right)^2.$$

The center of this circle is $x = \frac{s_3}{2s_2}$. Substituting for x in such a circle, we obtain

(17) $$\bar{x} = \frac{(s_2 - t)\,t}{s_2},$$

an expression which contains the first [illegible] of the extraneous circles of the family

directrix lines. If τ is the equation of a
plane, the two points where τ meets the
four bundle transversals are on a conic, and
it is easy to see that (11) gives the projective
law linearly on which the conic lies.

Having found this correlation
for four lines of an H^4 we can then locate
by the fundamental results for four lines of the
same —, and the rest of the lines by re-
solution of.

Take the equation of the H^4 in the
form: $t^6 - at^5 + \kappa t^4 - \mu t^3 + \bar{\kappa} t^2 - \bar{a} t + 1 = 0$
and that of the general H^4

(18) $\quad \kappa t^3 - u_1 t^2 + \kappa t - \alpha t^4 - \mu t^3 + t = 0$

Identify the coefficients t, t^2, t^3, t^4 and eliminate the
(twelfth) from the equation of the H^4. We have
(16') $\quad t^6 - a t^5 - t + 1 - (u - a)t^3 + t(\kappa t^2 - \alpha t^4 + t^6)$,
whose six roots $t_1, t_2, t_3, t_4, t_5, t_6$ give a set
of the common directrix lines of the two quartics

Now, divide (1) by t.

$$\gamma t^2 - \rho_1 t + \overline{\chi} - \alpha t^3 - \beta t^2 - \gamma - \frac{\delta}{t} = 0.$$

Differentiating as to t we obtain for χ

$$\chi = \frac{\delta_1}{2t^2} + \frac{3}{2}\alpha t + \beta - \frac{1}{2t^3},$$

the map equation of this general A^4 in this equation exists $k'+8$ for χ and the constant term 3 coefficients, that is, the ∞^3 is transformed to this point. Take this from 3 the number of the A^4. But from (18') 3 is obtained by ∞^2 for the 6 roots of this equation giving the ∞^2 cones of the whole. Also these six t's are from (18') connected by the relation $S_6 = 1$. Any five of these parameters give five lines determining thus an A^4 with β as center. As $\beta = S_2$ for 6 t's, it is equal to $S_2 + S_1 t_6$ for 5 t's. But $t_1 t_2 t_3 t_4 t_5 t_6 = 1$, that is, $t_6 = \frac{1}{t_1 t_2 t_3 t_4 t_5}$. Hence $\beta = S_2 + \frac{S_1}{t_1 t_2 t_3 t_4 t_5}$ (1) for 5 t's.

We perform in the same manner

of the A^6 is of the form

$$\chi t^2 - \bar{\chi} = k,$$

while the locus of the A^6 with parameter t_6 is

$$t_6^6 - a t_6^5 + \chi t_6^4 - \mu t_6^3 + \bar{\chi} t_6^2 - \bar{a} t_6 + 1 = 0$$

as before, if the perpendicular to the line $\ldots$ we make to obtain the locus of A^4 — which was $S_2 + \frac{S_1}{S_5}$ for five t's — we obtain

$$\chi t_6^2 - \bar{\chi} = t_6^2 \left(S_2 + \frac{S_1}{S_5}\right) - \left(\frac{S_3}{S_5} + S_4\right)$$

for which locus the equation is

$$(11) \quad \chi t^2 - \bar{\chi} = t^4 \left(S_2 - \ldots t + t^2 + \frac{(\omega - \eta)^2}{\omega}\right) - \frac{t(\ldots)}{\omega} - \left(S_4 - \ldots t + \ldots t^2 + t_6\right)$$

which on t having the values t_1 $\ldots$ reduces to the five perpendiculars, and each from the centre of the A^4 on σ, the locus of the A^2, upon which the same operation on each gives. (11) $\ldots$ simple form

$$\ldots \left(\chi - \frac{\omega}{\omega}\right) - \left(\bar{\chi} - \ldots t - \ldots\right),$$

the equation of a circle with center $\frac{S_2}{S_6}$ and radius $\left|\frac{S_3}{S_6} - \frac{S_3}{S_6}\right|$. Hence, we have the theorem that if from each center of the circumscribed A^4 of ∞^2 lines of direction, a pencil of indicatrices be laid off on the lines through it, the ∞^2 points of intersection are on a circle.

The equation

$$x = \frac{(S_2 - S_1 t + t^2)t}{S_7} \qquad S_7 \text{ for } 7\,t\text{'s}$$

gives the particular trochoidal curve on which are the centers of the seven curves, for a fixed t, like that given in Fig. 120.

I pass over the question as to what is the reciprocal of the orthocentric curves just considered. The first equation of (24) is $x = \frac{S_2}{S_6} + \frac{1}{2T}\left(S_3 - \frac{S_3}{S_6}\right)$

Symmetrizing for $2\infty^1$ on t's, we have

$$x = \frac{S_2 t - S_1 t^2 + t^3}{S_7} + \frac{1}{2T}\left[(S_3 - S_2 t + S_1 t^2 - t^3)\left(1 - \frac{t}{S_7}\right)\right],$$

which may be reduced to the form

(21) $\chi = \frac{s_1 t - s_2 t^2 + t^3}{\gamma} + \frac{1}{2T}\left[s_3 t^2 + s_4 t^3 + s_5 t^4 \right] - \frac{1}{\gamma}\left(s_6 - \frac{s_7}{t} + \frac{s_8}{t^2} - \frac{s_9}{t^3} \right)$

Now equation, when t takes the values t_i ($i = 1, 2, \ldots 7$) becomes the equation of the observations on the remaining six lines when t_i is omitted from the seven linear forms s_i in t: as in the previous case, to obtain the coefficients of these circles we must use $\frac{t \Delta_{ik}}{T \Delta_{7k}}$ = its conjugate. Differentiating (21) as to t and multiplying by γ we have

$$\gamma \delta_t \chi = s_1 - 2s_2 t + 3t^2 + \frac{1}{2T}\left[2s_3 t + 3s_4 t^2 \right] - \frac{1}{\gamma}\left(-\frac{s_7}{t^2} + \frac{2s_8}{t^3} - \frac{3s_9}{t^4} \right)$$

$$T \delta_T \chi = -\frac{1}{2T}\left[(s_3 - s_2 t + s_1 t^2 - t^3) - \frac{1}{s_7}\left(s_4 - \frac{s_5}{t} + \frac{s_6}{t^2} - \frac{s_7}{t^3} \right) \right]$$

$$t \delta_t \chi = (s_1 - \frac{2s_6}{t} + \frac{3s_7}{t^2}) \frac{1}{\gamma} + \frac{1}{2T}\left[\frac{1}{s_7}\left(-\frac{s_5}{t} + \frac{2s_6}{t^2} - \frac{3s_7}{t^3} \right) - (s_3 - 2s_2 t + 3s_1 t^2 - 4t^3) \right]$$

$$T \delta_T \chi = -\frac{1}{2}T\left[\left(\frac{s_4}{s_7} - \frac{s_5}{s_7 t} + \frac{s_6}{s_7 t^2} - \frac{1}{t^3} \right) - (s_3 - s_2 t + s_1 t^2 - t^3) \right]$$

Whence, subtracting we get as the equation of contact when t has the same value in both forms

$$\text{Case } \frac{(s_2 - 2s_1 t + 3t^2)t + \frac{1}{2T}\left[(-s_2 t + s_1 t^2 - t^3) - \frac{1}{s_7}\left(s_5 - \frac{s_6}{t} + \frac{s_7}{t^2} \right) \right]}{-\frac{1}{2T}\left[(s_3 - s_2 t + s_1 t^2 - t^3) - \frac{1}{s_7}\left(s_4 - \frac{s_5}{t} + \frac{s_6}{t^2} - \frac{s_7}{t^3} \right) \right]}$$

(22)

$$= \frac{\left(s_2 - \frac{2s_6}{t} + \frac{3s_7}{t^2} \right)\frac{1}{t} + \frac{1}{2}T\left[\frac{1}{s_7}\left(-\frac{s_5}{t} + \frac{2s_6}{t^2} - \frac{3}{t^3} \right) - (s_3 - 2s_2 t + 3s_1 t^2 - 4t^3) \right]}{-\frac{1}{2}T\left[\left(\frac{s_4}{s_7} - \frac{s_5}{s_7 t} + \frac{s_6}{s_7 t^2} - \frac{1}{t^3} \right) - (s_3 - s_2 t + s_1 t^2 - t^3) \right]}$$

The denominators of (21) have $-\frac{1}{t}$ as common, and the expressions in the [ ]'s are equal but opposite in sign. Hence it reduces to

$$(23)\quad \frac{(S_2 - 2S_1 t + 3t^2)tT}{S_7} + \frac{1}{tT}\left(S_5 - \frac{2S_6}{t} + \frac{3S_7}{t^2}\right)$$

$$-(S_2 t - 2S_1 t^2 + 3t^3) - \frac{1}{S_7}\left(\frac{S_5}{t} - \frac{2S_6}{t^2} + \frac{3S_7}{t^3}\right) = 0,$$

a quadratic in T whose roots are T and S_7. To obtain the other root, let

$$m = (S_2 t - 2S_1 t^2 + 3t^3)$$
$$n = \left(\frac{S_5}{t} - \frac{2S_6}{t^2} + \frac{3S_7}{t^3}\right),$$

Then (23) can be written as

$$\frac{1}{S_7} m T^2 - \left(m + \frac{n}{S_7}\right)T = -n$$

or

$$T^2 - \left(\frac{S_7 m + n}{m}\right)T = -\frac{S_7 n}{m},$$

and since the sum of the roots is $\frac{S_7 m + n}{m}$ and one root is S_7, the other root is $\frac{n}{m} \equiv \frac{S_5 t^2 - 2S_6 t + 3S_7}{t^4(S_2 - 2S_1 t + 3t^2)}$. The conjugate of this root is $\frac{t^4(S_2 - 2S_1 t + 3t^2)}{S_5 t^2 - 2S_6 t + 3S_7}$, that is, the second root is a linear. Thus both roots are linear.

Hence the real part of the envelope obtained by putting S_7 for T in (21). This

Substituting gives

$$X = \frac{S_3 t - S_1 t^2 + t^3}{S_7} + \frac{1}{S_7}\left[(S_3 - S_2 t + S_1 t^2 - t^3) - \frac{1}{S_7}\left(S_4 - \frac{S_5}{t} + \frac{S_6}{t^2} - \frac{S_7}{t^3}\right)\right]$$

or, simplifying,

$$X = \frac{S_3 + S_4 t - S_1 t^2 + t^3}{S_7} - \frac{1}{S_7^2}\left(S_4 - \frac{S_5}{t} + \frac{S_6}{t^2} - \frac{S_7}{t^3}\right).$$

Taking conjugate of this equation, we have

$$\bar{X} = \tfrac{1}{2}\left(S_4 + \frac{S_5}{t} + \frac{S_6}{t^2} + \frac{S_7}{t^3} - \frac{1}{2} S_7^2 \left(\frac{S_3 - S_2 t + S_1 t^2 - t^3}{S_7}\right)\right)$$

Multiplying this last equation by $\frac{1}{S_7^2}$ and subtracting the result from the former gives

$$X - \frac{1}{S_7^2}\bar{X} = \frac{S_3}{S_7} - \frac{S_4}{S_7^2}, \tag{24}$$

the equation of a line. Hence the theorem that the orthocentric circles of the 7 lines of derivation lie on a line.

In a similar manner we may derive the orthocentric circle for 8 derivation lines. Direct calculation gives as the equation of this circle:

$$t^2\left(X - \frac{S_3}{S_8}\right) - \left(\bar{X} - S_5\right) = t\left(S_4 - \frac{S_4}{S_8}\right), \tag{25}$$

and we have another theorem of the class in that if from each center of the con-

scribed on T_q's directed Circle of the A's perpendicular to the test face upon the Circle Light and Circumference, the X's perpendicular upon the Circle

The mark equation of this axis is

$$x = \frac{S_q}{S_4} + 2T\left(\lambda - \frac{S_5}{S_4}\right) \tag{26}$$

the parameter being T. This equation expressed and reduced gives

$$x = \frac{S_3 t - S_2 t^2 + S_1 t^3 - t^4}{S_4} + \frac{1}{2T}\left[\left(S_5 - S_3 t + S_2 t^2 - S_1 t^3 + t^4\right) \cdot \right.$$
$$\left. - \frac{1}{S_4}\left(S_5 - \frac{S_6}{t} + \frac{S_7}{t^2} - \frac{S_8}{t^3} + \frac{S_9}{t^4}\right)\right] \tag{27}$$

If now we make use of

$$\frac{T^2 x}{t} = \frac{}{T - x} \tag{28}$$

The get a quadratic whose two roots are

$$T_1 = S_4 \text{ and } T_2 = \frac{S_6 t^3 - 2 S_7 t^2 + 3 S_8 t - 4 S_9}{t^3 (S_3 - 2 S_2 t + 3 S_1 t^2 - 4 t^3)}, \tag{24}$$

the first of which put in (27) gives

$$x = \frac{S_4 + S_3 t - S_2 t^2 + S_1 t^3 - t^4}{2 S_4} - \frac{1}{2 S_4^2}\left(S_5 - \frac{S_6}{t} + \frac{S_7}{t^2} - \frac{S_8}{t^3} + \frac{S_9}{t^4}\right),$$

Furthermore, the complete [illegible] axis is

$$x - \frac{}{} = \frac{}{} - \frac{}{}. \tag{30}$$

[illegible]

of osculating conics of the 4-line to the 4-line are on a line.

Up to this point we have used for the metrical treatment of the projective 4-line the curve called the A^4, the 4-ic being considered a special case of the 7-line; for the 7-line we have used the curve A^7 the 6-ic being a special case of the 7-line; for the 7-line the curve A^7, the 7-ic being a particular case of the 7-line.

The general curve of this sum [illegible] for the metrical handling of $2n+1$ lines of direction. It is

$$(-)^n(\bar{x}t^{n+1} + \bar{x}t^{n-1}) = t^{2n+1} - (S_1 t^{2n-1} + S_{2n-1} t)$$
$$(11) \quad + \cdots\cdots + (-)^n S_n t^n + \cdots\cdots + (-)^n (S_{n+2} t^{n+2} + S_{n+2} t^{n-2})$$

This equation is the line form and the coefficients S_α and $S_{2n-\alpha}$ are conjugates. If

...nuous the chi integrals and therefore...
for a given k, there are $2n$ values for t,
that is, $2n$ tangents from a point P
to the curve. On substituting the value
of x and the n of equation (1) — $f = 0$
and the arbitrary λ in the equation
of a line $ax + \bar{a}\bar{x} + 1 = 0$, it is seen
that the curve is of order $2n+2$. By the
aid of this curve, which we may call
the H, and the H^{2n-2}, we can form
the general theorem of the two theorems
of theorems already indicated.

Hereafter when the H^{2n} on the form

(32) $t^{2n} - a_1 t^{2n-1} + \dots + (-)^{n+1} \chi t^{n+1} + (-)^{n} \mu t^{n} + (-)^{n+1} t^{n+1} + \dots \quad (-) \underset{2n-1}{\Delta} t + 1 = 0$

and also the general H^{2n-2} as form

(33) $\alpha t^{2n-2} - \beta t^{2n-3} + \gamma t^{2n-4} - \dots + (-)^{n} (\chi - \delta) t^{n} + (-)^{n+1} \mu_1 t^{n-1}$

$+ (-)^{n} [\chi - \lambda] t^{n-2} + \dots - \underset{2n-3}{\Delta} t^{2} + \dots - \underset{2n-2}{\Delta} t + \dots$

Combining these equations, we have —

$t^{2n} - a_1 t^{2n-1} + \dots - \underset{2n-1}{\Delta} t + 1 + [-]^{n} \lambda + [-]^{n+1} \mu_1] t^{n} +$

$+t\left[\alpha t^{2n-2} - \beta t^{2n-3} + \gamma t^{2n-4} \cdots \mp \delta t^{n+1} \mp \lambda t^{n-1} + \cdots - \lambda t + \theta\right] - \zeta$,

an equation whose $2n$ roots are the n reciprocals of $2n$ consecutive diagonals of the H^{2n} and H^{2n-1}. The product of the H^{2n-1}

$$S = \pm \tau_1 \tau_2 \tau_3 \cdots \tau_{n-1} = \mp \nu_{n-1},$$

because on eliminating the n quantities the term containing S is the only one containing t, when the equation is explicitly solved for t. Here τ is connected with the definition of preceding H and ν_{n-1} have to be so to $2n+1$ connected by the relation $\nu_{2n} = 1$. The H^{2n+1} in passing $2n-1$ lines is (34) $X = \nu_{n-1} + \frac{\nu_{2n-1}}{\tau_{2n-1}}$. and it follows that S's are for $2n-1$. So before in the preceding theorem we considered that H for n reduction to a cubic if the H^{2n} were on the form $\nu_{n-1} + \frac{\nu_{2n-1}}{\tau_{2n-1}}$. This

remark) the A^{2n-1} in $2n-1$ [illegible], so

$$t_{2n}^2 X - X \quad t_{2n}^2\left(S_{2n-1} + \frac{S_{2n-2}}{S_{2n-1}}\right) - \left(\frac{S_{2n}}{S_{2n-1}} + S_{2n+1}\right).$$

This [illegible] for $2n+1$ this

[illegible]

(35) $$t^2 X - \frac{S_{2n-1}}{S_{2n}} X - S_{2n+1} = t\left(S_{2n} - \frac{S_{2n}}{S_{2n}}\right),$$

this, the centre circle for $2n$ circles, we can state precisely, therefore, that if from each centre of this A^{2n-2} on $2n-1$ of $2n$ tangent circles of A^{2n}, a perpendicular be laid off on the common tangent at each point, then is perpendicular [illegible] equal.

The point equation of this circle is

(36) $$X = \frac{S_{2n-1}}{S_{2n}} + \frac{1}{2T}\left(S_{2n} - \frac{S_{2n}}{S_{2n}}\right),$$

which, symmetrized for $2n+1$ t's, gives

(37) $$X = \frac{S_{2n-1}t - S_{2n-2}t^2 + \cdots + (-)^{2n-1}t^{2n}}{S_{2n+1}}$$

$$+ \frac{1}{2T}\left[S_{2n} - S_{2n-1}t + \cdots + (-)^{2n}t^{2n} - \frac{1}{S_{2n+1}}\left(S_{2n+1} - \frac{S_{2n+2}}{t} + \cdots + (-)^{2n}\frac{S_{2n+1}}{t^{2n}}\right)\right]$$

an equation, which for $t = t_1, t_{2n+1}$ reduces to the respective [illegible]

2n if 2n+1 terms. Without it the ... of ... ; the equation of condition for ... that

$$\left(\frac{1}{t} - \frac{S_{n-1}}{S_{2n+1}}t + \cdots + (-)^{n+1}\frac{n t^{n-1}}{S_{2n+1}}\right)T + \frac{1}{2T}\left[\left(-S_{n-1}t + \cdots (-)^n t^n - \frac{1}{S_{2n+1}}\left(\frac{S_{n+2}}{t} - \cdots (-)^n \frac{n S_{2n+1}}{t^n}\right)\right.\right.$$

$$\left.\left. - \frac{1}{2T}\left[\left(S_n - S_{n-1}t + \cdots + (-)^n t^n\right) - \frac{1}{S_{2n+1}}\left(S^{n+1} - \frac{S_{n+2}}{t} + \cdots - (-)^n \frac{S_{2n+1}}{t^n}\right)\right]\right.\right.$$

$$\left.\left(S_{n+2} - \frac{S_{n+2}}{t} + \cdots + (-)^{n+1}\frac{n S_{2n+1}}{t^{n-1}}\right) + \frac{1}{2}T\left[\frac{1}{S_{2n+1}}\left(-\frac{S_{n+2}}{t} + \cdots (-)^n \frac{n}{t^n}\right) - \left(S_{n-1}t - \cdots (-)^{n+1} n t^n\right)\right]\right.$$

$$= -\frac{1}{2}T\left[\left(\frac{S_{n+1}}{S_{2n+1}} - \frac{S_{n+2}}{S_{2n+1}t} + \cdots + (-)^n \frac{1}{t^n}\right) - \left(S_n - S_{n-1}t + \cdots + (-)^n t^n\right)\right]$$

which reduces to

$$\frac{\left(S_{n-1} - 2S_{n-2}t + \cdots + (-)^{n+1} n t^{n-1}\right)tT}{S_{2n+1}} + \frac{1}{tT}\left(S_{n+2} - \frac{2S_{2n}}{t} + \cdots + (-)^{n+1}\frac{n S_{2n+1}}{t^{n-1}}\right)$$

(38)

$$-\left(S_{n-1}t - 2S_n t^2 + \cdots + (-)^{n+1} n t^n\right) - \frac{1}{S_{2n+1}}\left(\frac{S_{n+2}}{t} - \frac{2S_{2n}}{t^2} + \cdots + (-)^{n+1}\frac{n S_{2n+1}}{t^n}\right) = 0$$

whose two roots are $T_1 = S_{2n+1}$ and

$$T_2 = \frac{S_{n+2}t^{n-1} - 2S_{2n}t^{n-2} + \cdots + (-)^{n+1} n\left(S_{2n+1}\right)}{t^{n+1}\left(S_{n-1} - 2S_{n-2}t + \cdots + (-)^{n+1} n t^{n-1}\right)}$$

both of which are known. $T = S_{2n+1}$ ... (39)

gives $\chi = \frac{S_n + S_{n-1}t - \cdots + (-)^{n+1}t^n}{2S_{2n+1}} - \frac{1}{2S_{2n+1}}\left(S_{n+1} - \frac{S_{n+2}}{t} + \cdots + (-)^n \frac{S_{2n+1}}{t^n}\right)$.

This combined with its conjugate results in

$$(3\cdot1) \qquad \chi - \frac{1}{S_{2n+1}^2}\overline{\chi} = \frac{S_n}{S_{2n+1}} - \frac{S_{n+1}}{S_{2n+1}^2},$$

the equation of a line. We thus have the enumerating theorem of the chain or lines that the $2n+1$ orthocentric circles of $2n+1$ tetrastichal lines are on a line.

The results obtained thus far may be summarized as follows:

(a) The four perpendiculars let fall each from the center of the circle of 3 of 4 directed lines upon the line left out are on a circle, the orthocentric circle for the 4-line.

(b) The five orthocentric circles of 4 of 5 directed lines are on a line, the locus of the 5-line.

(c) The six perpendiculars let fall each from the center of the A^4 for 5 of 6 lines upon the

omitted line are on a circle.

(d) The seven orthocentric circles for 6 of 7 lines are on a line.

(e) The eight-perpendiculars let fall each from the center of the A^6 of 7 of 8 lines upon the line left out are on a circle.

(f) The nine orthocentric circles for 7 of 8 lines are on a circle.

(g) Generally: the $2n$ perpendiculars let fall each from the center of the A^{2n-2} on $2n-1$ of $2n$ lines on the middle of line are on a circle.

(h) Generally: the $2n+1$ orthocentric circles of $2n$ of $2n+1$ lines are on a line.

The equation for (a) ---- (h) follows, the circles and lines being taken for reference by

(α) $(x-\frac{S_1}{S_4})t^2-(\bar{x}-S_3)=t(S_2-\frac{S_2}{S_4})$ $2n=4$

(β) $(x-\frac{S_2}{S_6})t^2-(\bar{x}-S_4)=t(S_3-\frac{S_4}{S_6})$ $2n=6$

(γ) $(x-\frac{S_3}{S_8})t^2-(\bar{x}-S_5)=t(S_4-\frac{S_6}{S_8})$ $2n=8$

(δ) $(x-\frac{S_{n-1}}{S_{2n}})t^2-(\bar{x}-S_{n+1})=t(S_n-\frac{S_n}{S_{2n+1}})$ $2n=2n$

$$(2') \quad \chi - \frac{1}{\frac{1}{c}}\bar{\chi} = \frac{\frac{1}{c}}{\frac{1}{c}} - \frac{\frac{1}{c}}{\frac{1}{c}} \qquad \text{[illegible]}$$

$$(3') \quad \chi - \frac{1}{\frac{1}{\eta}}\bar{\chi} = \frac{\frac{1}{\eta}}{\frac{1}{\eta}} - \frac{\frac{1}{\eta}}{\frac{1}{\eta}} \qquad \text{[illegible]}$$

$$(4') \quad \chi - \frac{1}{\frac{1}{\gamma}}\bar{\chi} = \frac{\frac{1}{\gamma}}{\frac{1}{\gamma}} - \frac{\frac{1}{\gamma}}{\frac{1}{\gamma}} \qquad \text{[illegible]}$$

$$(5') \quad \chi - \frac{1}{\frac{1}{\gamma_{n+1}}}\bar{\chi} = \frac{\frac{1}{\gamma_{n+1}}}{\frac{1}{\gamma_{n+1}}} - \frac{\frac{1}{\gamma_{n+1}}}{\frac{1}{\gamma_{n+1}}} \qquad \text{[illegible]}$$

Hence for any finite number n of terms.

If γ_{n+1} tends to zero, $\frac{1}{\gamma_{n+1}}$ tends to infinity; for any finite value of χ, $\frac{\chi}{\gamma_{n+1}}$ tends to zero, and $\frac{1}{\gamma_{n+1}}$ tends to infinity.

Equation (17) is a fraction of the form $\frac{0}{0}$ when $\chi = \frac{1}{\gamma_{n+1}}$, and $\frac{\infty}{\infty}$ when $\chi = \frac{1}{0}$. Hence χ has a particular value, which is the limit of the fraction when χ tends to $\frac{1}{\gamma_{n+1}}$, and another particular value when χ tends to $\frac{1}{0}$. The former is in general finite, but the latter is infinite.

of circles in (b), is

$$k = \frac{(S_{n-1} - S_{n-2} + \cdots \pm 1)\,t}{2n+1},$$

an octavic; and that for (g) it is

$$k = \frac{(S_{n-1} - S_{n-2} + \cdots \pm 1)\,t}{2n+1}$$

a 2n-ic.

Section 2.

Extension of the Theorem that the In-centers of 3 of 4 Lines are on a Circle.

The Chain of Equidistance Theorems.

The proof was given of the theorem that the center of the A^4 is equally distant from the centers of the circle on the 4 in centers and from the center of the circle touched by the 4 Escribed circles from the in centers of 3 of 4 lines on the same plane.

It will be shown that this is the first of a chain of similar theorems. A theorem will first be given relating to the circle touched by the 4 Escribed circles and the 4 circumscribed circles of 3 of 4 lines

use in a circle. Hence we, if necessary, [illegible] possible if the [illegible] 3 of them determined a circle which is also in the $A^{(2)}$.

The result of the A^4 [illegible] lines of the A^4 and A^3 are

$$k = S_2 + \frac{1}{S_1}$$

where the S' are for $t = t'_1$. By a similar [illegible] for a [illegible], we have

$$k = S_2 + t + t^{-1} + \frac{(S_1 - t)t}{S_2}, \tag{4 U}$$

that expression of a [illegible] the [illegible] S_2. Hence, we have the same [illegible] if the change in beginning with the number of the incircles for $3 \cdot 1 - 4$ terms, viz., that the [illegible] number of the [illegible] A^4's of S of a [illegible] of the A^4 has on a [illegible] which [illegible] is S_2. As first possible, in general, determine a [illegible], there is [illegible] not any 6 possible, but are connected by the [illegible] [illegible].

But the absolute value of the ratio giving the center of the orthocentric circle for which $|z| = 1$. Hence $\left|\frac{s_2}{s_6}\right| = |s_6|$, that is, we have the theorem that the center of the A^6 which is known and given is opposite, distant from the center of the Lemoine circle the center of the 6 A^4's or 5 of 6 common lines of the A^4 and A^6 and the center of the orthocentric circle of the 6 lines. This is also a generalization of the Feuerbach theorem. Moreover, this proves the distance theorem shows that the center of the A^4 on 5 lines is given as the mean of the 5 perpendicular bisectors, one each for the pairs of the 5 lines and orthocentric circle for the 5 lines of 4 lines in class 5.

And a similar statement is true in the present case that is to be

The center of the A^4 corresponding to the pair of lines is given as the point of the I hyperboloid of which the w of each point of the conic of the orthogonal pair of lines of I and those of the second kind for the same lines.

Equating, the center of each of the ∞^1 common lines of the A^{10} and A^8 was the point $\qquad x = s_3 + \frac{s_2}{s_7}$,

and that for each of these lines is

$$x = s_3 - s_2 t + s_1 t^2 - t^3 + \frac{(s_2 - s_1 t + t^2)t}{s_8} \tag{11}$$

that is, the ∞^1 centers of the A^4's of the ∞^1 lines are connected by the relation that they are on the w plane of each point of the conic in the plane $\omega = s_3$; and since $|s_2|$ is greater than $|s_1|$ and $|s_3|$ [illegible] of hyperboloids [illegible]. Hence, again, the theorem that the centers of the A^4's which [illegible]

from the center of the curve of centers of A^{n-1}, on 7 of 8 lines and the center of the other conic coincides for 7 lines.

These theorems can be generalized, the first, that the $2n$ centers of the A^{n-2}, on $2n-1$ of $2n$ lines of direction are on the conic

$$(+1) \quad K = S_{n-1} - S_{n-2}t + \cdots + (-1)^{n+1}t^{n-1} + \frac{(S_{n-1} - S_{n-3}t + \cdots + (-1)^{n-1}t^{n-1})t}{S_{2n}}$$

Also, since the center of the other conic coincides for $2n$ lines if $\frac{S_{n-1}}{S_{2n}}$ is the center of the curve for which S_{n-1}, we have

Generally, that the centers of the A^{n-2} of which $2n$ lines are given as equally [illegible] from the centers of the A^{n-1} [illegible] $2n-1$ of $2n$ lines

Section 3.

The Curve A^{2n}.

The general curve of the same type as the A^4 curve A_1^4 of class 3 and the 2 11+ 4 has been already given in the form

$$(-)^n(\lambda t^{n+1} + \bar{\lambda} t^{n-1}) = t^{2n} + 1 - (s_1 t^{2n-1} + \sum_{2n-1} t) + (-)^n \mu t^n$$
$$+ - \quad + (-)^{n-1}\left(\sum_{n-1} t^{n+1} + \sum_{n+1} t^{n-1}\right).$$

Consider the case of $n = 3$. This gives a curve A_3^6 viz.,

$$t^6 - s_1 t^5 + \lambda t^4 - \mu t^3 + \bar{\lambda} t^2 - s_5 t + 1 = 0$$

The map equation is

$$(43) \qquad \lambda = -2t^2 + \tfrac{3}{2} s_1 t + \frac{\mu}{2t} - \frac{s_5}{2t^2} + \frac{1}{t^4}$$

Differentiate this map equation and get the reciprocal equation in t whose roots are the parameters for cusps of the curve:

$$0 = -4t + 3s_1 - \frac{\mu}{t^2} + \frac{2s_5}{t^3} - \frac{2}{t^5},$$

etc. etc., that we shall not give here.

(44) $\quad 8(t^6+1) = 3(S_1 t^5 + S_3 t) - \mu t^3$

Hence, [illegible],

of the A^6 with the cusp condition (44), we have $-\delta(\kappa t^4 + \bar{\kappa} t^4) = \varepsilon(t^6+1) - \delta(S_1 t^5 + S_3 t) - \delta\mu t^3$

$0 = 8(t^6+1) - 3(S_1 t^5 + S_3 t) + \mu t^3$

that is, (45) $\quad \delta(\kappa t^3 + \bar{\kappa} t) = -\varepsilon(S_1 t^4 + S_5) + q\mu t^4$

But (45) is the equation of an A^4 concentric with A^6. Hence the theorem that the [illegible] a concentric A^4.

This theorem may be extended to the t^{2n}. The [illegible] of the A^{2n}

(46) $(-)^n(t^{2n}+1) = \left[(n+1)t^{n-1} - \frac{n-1}{1}\right] - \left[\frac{(n-1)}{1}t^{n-2} - \frac{(n-2)}{1\cdot 2}\right] + \cdots \frac{t^n}{n} + \cdots + (-)^n\left(\frac{n}{n-1}t - \frac{n}{n}\right)$

Differentiating [illegible], on reduction the cusp condition,

(47) $0 = (n+1)(n-1)(t^{2n}+1) - \left[S_1 n(n-1)t^{2n-1} + S_{2n-1}\, n(n-1)t\right]$

$+ \cdots + (-)^n \mu t^{2n} + \cdots + (-)^n\left[3S_{n-1}t^{n+1} + 3S_{n+1}t^{n-1}\right]$

Recombining now (47), and [illegible]

arranging, we have

$$(+8)\ (-)^n n(n+1)(n+2)\left[\frac{1}{2}t^{n}+\frac{1}{2}t^{n-1}\right]=\left[\frac{1}{2}t^{n+2}+\frac{1}{2}t^{n+1}\right]$$
$$+\cdots+(-)^{n}\frac{n^2}{2}t^{n-1}+\cdots+(-)^{n}\frac{(n^2-2)}{2}\left[\frac{1}{2}t^{n+1}+\frac{1}{2}t^{n}\right].$$

But (4 8) is the equation of an A^{2n-2} concentric with the A^{2n} given; hence the theorem that there are $2n$ such tangents from t^n and t^{n+1} and hence angle touch a concentric A^{2n-2}.

Section 14.

Orthocentric Properties of the Curve of a Directed $2n$-Locus

Consider any four lines of the plane, and let them be cuts of the curve A^4, already defined, which is uniquely determined by 5 lines. The centre of the orthocentric circle for the four lines where parameters are t_i $(i=1,2,3,4)$ was found to be $\frac{S_1}{S_4}$.

$\frac{S_1}{4}$ for four t's is equal to $\frac{S_1 + t_4}{S_3 t_4}$ where the S's are for 3 t's. Obviously, if S_1 for 3 t's in the latter expression is zero, that is, if

$$t_1 + t_2 + t_3 = 0,$$

then the center of the orthocentric circle is independent of the 4th line, since in that case $\frac{S_1 + t_4}{S_3 t_4} = \frac{1}{S_3}$ (S_3 for 3 t's).

If $t_1 + t_2 + t_3 = 0$, so is its conjugate $\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} = 0$, and the values of the parameters t_1, t_2, t_3 are respectively

$$t_1, \omega t_1, \omega^2 t_1$$

where ω refers to the cube roots of unity. In this last case the directrices of the 3 lines are

$$-\frac{1}{t_1^2}, -\frac{1}{\omega t_1^2}, -\frac{1}{\omega^2 t_1^2}$$

and the lines themselves form an equilateral triangle. Further if we combine the equations of two lines of the 4th with parameters t_1 and $\omega^2 t_1$, we get

$$t_1^4 - 2t_1^3 + 6t_1^2 - 2t_1 + 1 = 0$$

and $\omega^2 t_1^4 - k_1 t_1^3 + \mu_1 t_1^2 \omega - \omega k t_1 + 1 =$

and obtain

$$(1-\omega^2) t_1^4 + (1-\omega)\mu t_1^2 - (1-\omega^2) k t = 0$$

or $$k_{1,2} = t_1^3 - \mu\omega t_1 ,$$

the conjugate of the meet of the two lines.

Similarly, the two other values are

$$k_{1,3} = t_1^3 - \mu t_1 \omega^2$$

$$k_{2,3} = t_1^3 - \mu t_1$$

Adding and dividing by 3, one has

$$k = \frac{1}{t_1^3} ,$$

that is, the invariant of three points is

$$\frac{1}{t_1^3} ;$$

but, in the expression $\frac{t_1 + t_4}{t_1 t_4}$, t_1 is any, then the expression is $\frac{1}{t_1^3}$, whence

[illegible]

centric circle is the centroid of the 3 points of intersections of the lines,

Take next the case of 6 lines tangent to an A^6, which is itself uniquely determined by seven lines. The orthocentric circle $=0$, for 6 lines,

$$t^2(x-\tfrac{S_1}{S_6})-(\bar{x}-S_4)=t(S_3-\tfrac{S_2}{6}),$$

The center is, then, $\frac{S_2}{S_6}$ for 6 lines. For 5 lines it is $\frac{S_2+\cdots}{5+\cdots}$. From what we have just found about the center of the osculant of the 6th line when $S_6=0$ it is easy, and also to be evident from symmetry $\frac{S_1}{6}$.

If for 5 lines $S_1=0$, the equation $\bar{x}_3=c$, and hence the lines for an A^5 are an axis for a pencil, the focal line, and the orthocentric circle for a system of 3 lines of the pencil. For 4 lines

we again get a function of t, and in that case the center of the circle is

$\frac{S_1}{S_5}$, S's for 5 t's.

We can not make the other conditions for 6 lines independent of two of the circles, so we have the four equations

$t_1 + t_2 - t_3 - t_4 = 0$

and the others are $\frac{1}{t_1} - \frac{1}{t_2} + \frac{1}{t_3} - \frac{1}{t_4} = 0$,

showing that the circles have the same property that the hyperbolas had.

For the center of 7 circles, the center of the orthogonal circle was

$\frac{S_2}{S_1}$

$\frac{S_3}{S_2}$ for 5 t's, i.e. $\frac{S_3 + S_2' t_5}{S_2 + S_1' t_5}$ for 7 t's from which it is evident that the center of the circle is independent of the 1st line since S_3 for 7 t's is zero. Then the center is

$\frac{S_2}{S_1}$ for 7 t's,

but $\frac{S_2}{S_1}$ for 7 t's is $\frac{S_2' + S_1' t_1}{S_2 + S_1' t_1}$ for 6 t's, the

center of this circle will be independent of the tth locus when t is for t's in place, taking them this number is

$$\frac{S_1}{S_6} \quad \text{for } t \text{'s}$$

$\frac{1}{S_6}$ for t t' is $\frac{S_1 + t_6}{S_5 t_6}$ for 5 t's. If S_1 for 5 t's is zero, the center of the circle is $\frac{1}{S_5}$ for 5 t's,

If S_1 for 5 t's is zero, we use the homogeneous $\frac{S_4}{S_5} = 0$, or $S_4 = 0$, $T_{6,1}$ then, t_6 for 6 t's is

$$T_1 + t_6 \quad \text{for } 6 \ t\text{'s}.$$

Look for one satisfying to the locus S_1 for 5 t's is zero, otherwise T_1 for 5 t's is zero, and if t_6 is zero, we will t_6. Otherwise we have for 5 t's

$$S_1 = 0$$
$$S_2 = 0$$
$$S_3 = 0$$
$$S_4 = 0,$$

for in which case four of the 5 parameters for the 5 circles have only two

Summing relations

$$\xi_1 - \xi_2 = \xi_3 + \xi_4 - \xi_5 = \xi_6 + \xi_7 - \xi_8 = \dots,$$

where in the nth (fifth mode of vibration). The nth lines thus have elements respectively equal to $-\frac{1}{\xi_1}, -\frac{1}{\xi_2}, -\frac{1}{\xi_3}, \frac{1}{\xi_4}, -\frac{1}{\xi_5}$, that is, the lines form an hyperbolic envelope. In taking the nth lines passing the points of inflexion of the curves other than the nth, it can be shown that the number of these intersections is $-\frac{1}{\xi_n}$; but this is the number of the other curves for which ξ_n is independent of the $1^{st}, 2^{nd}$ and 3rd terms, we have just seen. Hence, the number of the other curves for which ξ_n lies between A_n and A_{n+1} is $-\frac{1}{\xi_n}$. If we reject with the $4^{th}, 7^{th}, 8^{th}$ curves all but the corresponding curves form an open envelope as the

of the $\frac{n!}{2}$ circles that can be drawn by the $\frac{n!}{2}$ permutations of the lines for angles $\frac{\pi}{n}$ is the center of the orthocentroidal circle.

In regard to the 10 lines of an A^{10}, if we attempt to make the center of the orthocentroidal circle independent of the l^{10}, q^{10}, r^{10}, and s^{10}, the remaining lines will form a T-group involving p and t only. But we can make the center independent of p by the same method that we used in the case of the A^9, and we may state generally that, n being even, the center of the orthocentroidal circle is independent of $n+1$ of the lines when the remaining $n+1$ lines form an $n+1$-gon with angles $\frac{\pi}{n+1}$, and the centroid of the pseudo-n intersections when the angles are $\frac{2\pi}{n+1}$ is the center of the circle for the n lines. If n is odd, the result is independent of n of the lines when the remaining lines form an $n+2$-gon, and the center of the circle is $\frac{5!}{n+1}$, as above when n is even for $n+2$ lines.

Part II.

Derivation of the Equations of the Orthocentric Circle and Directrix by the use of the "Characteristic" constants of Directed Lines. Change of Direction of Lines.

Section I.

Derivation of the Equations.

The theorems of Part I. on orthocentric circles and directrices can be proved by the use of the so-called characteristic constants of directed lines, as employed by Love in the paper referred to before*.

As we are pointing out in another place, the equation of a straight line adapted to the study of lines of direction is

$$(49) \qquad x t_1 - 2r_1 + \frac{\bar{x}}{t_1} = 0$$

r being r cos, and the distance of the line from the origin, as is evident from putting in the formula for distance from point to line 0 for x_0 and $\bar{x}_0$, that is,

$$\frac{t x_0 - 2r + \frac{x_0}{t_0}}{-2\sqrt{t \times \frac{1}{t}}} = \text{distance} = r.$$

Also, adopting the definition of direction of lines as explained on page 6, there

See page 4 of this paper.

belongs to 2 lines to which directions have been arbitrarily assigned — one of the equidistant points, viz., the center of the inscribed circle of the triangle of the lines, or one of the 3 escribed centers, according to the assigned directions of the lines.

This equidistant point is found as follows: Denote the lines by the figures 1, 2, 3 and their respective parameters by t_1, t_2, t_3. Then the locus of equidistant points for lines 1 and 2 is

$$\left(x t_1 - 2 r_1 + \frac{\bar{r}}{t_1}\right) - \left(x t_2 - 2 r_2 + \frac{\bar{r}}{t_2}\right) = 0$$

in which the expressions in the form therein are taken positive, their values being the tangents of the supplementary angles at the intersection of the lines 1 and 2. Similarly,

$$(50) \qquad \left(x t_1 - 2 r_1 + \frac{\bar{r}}{t_1}\right) - \left(x t_3 - 2 r_3 + \frac{\bar{r}}{t_3}\right) = 0$$

represents the equidistant axis of lines 1,

case 3. Eliminating $\overline{x}$, we find

$$x = \frac{2r_1 t_1}{(t_1-t_2)(t_1-t_3} + \frac{2r_2 t_2}{(t_2-t_1)(t_3-t_1} + \frac{2r_3 t_3}{(t_3-t_1)(t_3-t_2)}$$

or (5-1) $$x = \sum^{3} \frac{2r_1 t_1}{(t_1-t_2)(t_1-t_3)}$$

Each value of t gives the equidistant point of the lines 1, 2, 3. Call the value of x a_1, one of the characteristic constants of the 3-lines. Symmetrizing, we have

(5-2) $$x = \sum^{4} \frac{2r_1 t_1 (t_1-t)}{(t_1-t_2)(t_1-t_3)(t_1-t_4)}$$

which is form $x = a_1 - a_2 t$ the equation of the circle on the 4 equidistant points of 3 of 4 lines. This is the Envelope of the 4 lines. In (5-2)

$$a_1 = \sum^{4} \frac{2r_1 t_1^2}{(t_1-t_2)(t_1-t_3)(t_1-t_4)} = \sum^{4} \frac{2r_1 t_1^{4-1-1}}{(t_1-t_2)(t_1-t_3)(t_1-t_4)}$$

and $$a_2 = \sum^{4} \frac{2r_1 t_1}{(t_1-t_2)(t_1-t_3)(t_1-t_4)} = \sum^{4} \frac{2r_1 t_1^{4-2-1}}{(t_1-t_2)(t_1-t_3)(t_1-t_4)}.$$

Thus, in general a_x for n lines is

$$(3\cdot 3)\qquad \sum \frac{2r_1 t_1^{2n-\alpha-1}}{(t_1-t_2)(t_1-t_3)\cdots\cdots(t_1-\overline{t_{2n}})}$$

where conjugate sets

$$(3\cdot 4)\qquad \overline{\sigma_\alpha} = (-)^{\alpha-1} S_{2n-\alpha} \qquad (S = t_1 t_2 t_3 \cdots t_{2n})$$

Hence σ's may be called the elementary symmetric functions of the conjugates.

With this preliminary work, we may proceed to the proof of the theorem of theorem of Prof. Cayley.

Let us, then, take as the value of the sum Σ the expression

$$\frac{r_1}{t_1-t_2} + \frac{r_2}{t_2-t_3} + \cdots + \frac{r_{2n}}{t_{2n}-t_1}$$

Suppose that the α roots of the equation $\phi = 0$ can be divided into β systems, each containing γ roots, so that $\alpha = \beta\gamma$, and that the sum of the γ roots of each system is x_1. The element η becomes ∞ as $\frac{1}{t_1-t_2}$, that is, as $\frac{1}{t_1-\overline{t_1}}$; and the function Σ finally is

$$\infty \cdot \frac{r_1}{t_1-\overline{t_1}} = \text{a root}.$$

$$1 - \frac{\bar{x}}{t_4^2} = \sum \frac{2r_1 t_1}{(t_1-t_2)(t_1-t_3)} - \frac{s_3}{t_4^2} \sum \frac{2r_1}{(t_1-t_2)(t_1-t_3)}$$

$$1 - \frac{\bar{x}}{t^2} = \sum \frac{2r_1 t_1(t_1-t)}{(t_1-t_2)(t_1-t_3)(t_1-t_4)} - \frac{s_4}{t^2} \sum \frac{2r_1(t_1-t)}{(t_1-t_2)(t_1-t_3)(t_1-t_4)}$$

when t takes successively the values t_4, t_3, t_2, t_1.

In this way is found the equation

$$(5\text{-}5') \qquad 1 - \frac{\bar{x}}{t^2} = a_1 - a_2 t - \frac{s_4}{t^3}(a_2 - a_3 t)$$

or

$$t^4(1 - a_1) - t^3(\bar{x} + s_4 a_3) = a_2(-t^3 - s_4 t),$$

from which follows for $\bar{x}$ the equation

$$-t^3 - \frac{s_4}{t} = -s_1 t^2 + s_2 t - s_3,$$

which on substituting for $\bar{x} - t^2$ its value in (5-5'), gives

$$(5\text{-}6) \qquad t^4(1 - a_1) - t^3(\bar{x} + s_4 a_3) = a_2(-s_1 t^2 + s_2 t - s_3),$$

The circle on the 4 perpendiculars

[illegible] from each [illegible]

Her fremit ligningene for $\dot{x}$ blir nu (1):

$$\dot{x} = \dot{x}_1 - \dot{x}_1 t + \frac{1}{2}\ddot{x}_1 t^2 .$$

Disse må nå settes sammen med de gitte $\dot{x}$-verdier. Vi får så ligningene for å bestemme $\dot{x}_1$, $\ddot{x}_1$ og $\dddot{x}_1$, hvorav de to siste kommer til å bli overbestemte. Vi får

$$\dot{x}_2 - \dot{x}_1 = \dot{x}_1 t_2 + \frac{1}{2}\ddot{x}_1 t_2^2 + \frac{\alpha_2 - \alpha_1 t_2}{2}\dddot{x}_1 t_2^2$$

$$-\frac{\alpha_1 - \alpha_3 t_3}{3}\left(\dddot{x}_1 t_3^3 + \frac{3}{2}\ddot{x}_1 t_3^2 + \dot{x}_1 t_3 + \dot{x}_1 - \dot{x}_3\right)$$

Dersom vi nu skal ha samme $\dot{x}$-verdi for $t = t_3$ uansett om vi bruker den ene eller andre av disse to $\dot{x}$-formlene, må vi ha

$$\dot{x}_2 - \dot{x}_3 = \dot{x}_1 t_2 + \frac{1}{2}\ddot{x}_1 t_2^2 + \frac{\alpha_2 - \alpha_1 t_2}{2}\dddot{x}_1 t_2^2 - \dot{x}_1 t_3 - \frac{1}{2}\ddot{x}_1 t_3^2 - \frac{\alpha_1 - \alpha_3 t_3}{3}\dddot{x}_1 t_3^3 .$$

Vi setter $t_2 - t_3 = \delta_{23}$, $t_1 - t_3 = \delta_{13}$, $t_1 - t_2 = \delta_{12}$. Videre $\frac{\dot{x}_2 - \dot{x}_3}{t_2 - t_3} = \frac{\delta_{23}}{\delta_{23}}$, $\frac{\dot{x}_1 - \dot{x}_3}{t_1 - t_3} = \frac{\delta_{13}}{\delta_{13}}$.

Vi skal også ha at $\frac{\dot{x}_1 - \dot{x}_2}{t_1 - t_2} = \frac{\delta_{12}}{\delta_{12}}$ osv. Da blir

$$\frac{\delta_{23}}{\delta_{23}} = \frac{\delta_{13}}{\delta_{13}} - \frac{\delta_{12}}{\delta_{12}} .$$

Vi setter så $\frac{\alpha_2 - \alpha_1 t_2}{2} = \alpha_4$ og $\frac{\alpha_1 - \alpha_3 t_3}{3} = \alpha_5$.

Vi får så ligningene for å bestemme $\ddot{x}_1$ og $\dddot{x}_1$, hvorav de to siste kommer til å bli overbestemte. Vi får

$$\frac{\delta_{23}}{\delta_{23}} t - \dot{x}_3 = \frac{\alpha_1}{\alpha_3}\dot{x}_1 - \frac{\alpha_2}{\alpha_3}\ddot{x}_1 t + \frac{\alpha_3}{t}\dddot{x}_1 - \alpha_4 \delta_{23} -$$

$$-(\alpha_1 - \alpha_3 t)\left(\frac{\delta_{12}}{\alpha_3} + \frac{\alpha_2}{\alpha_3}\dddot{x}_1 t - \frac{\alpha_2}{\alpha_3}\dddot{x}_1 t_3 + \frac{\alpha_3}{t}\dddot{x}_1 - \frac{\alpha_3}{t}\dddot{x}_1 t_3\right)$$

som vi kan skrive $\frac{\delta_4}{t} - \frac{\delta_5}{t^2}$ som en hensiktsmessig form

$$= t^2 \alpha_4 + t\alpha_5 + \delta_{23} .$$

Expanding we have,

$$\frac{a_2^2}{a_3^2}x - \bar{x} = \frac{a_1 a_2^2}{a_3^2} - \frac{a_2^2}{a_3^2}t + \frac{a_1 s_3}{t} - a_4 s_3 - \frac{s_1 a_2^2}{a_3^2}$$

$$- \frac{a_2^2}{a_3}(s_1 t - t^2) - \frac{a_2 s_4}{t} + \frac{a_2 s_3}{t^2} + \frac{a_2^2}{a_3}(s_1 t + \ldots) - \frac{1}{2}s_1^2 t$$

$$\ldots s_1 t^2 - t^3)a_2 + a_3 a_4 - \frac{a_3 s_3}{t^2} + \frac{a_2^2}{a_3^2} - t .$$

In reducing this equation we notice that the terms $\frac{a_2^2}{a_3^2}t$, $\frac{a_2 s_3}{t}$, $\frac{a_2^2}{a_3}(s_1 t - t^2)$ do not contain the [illegible] sign in the equation. Also the expression $(s_1 t - t^2)a_2 - a_2 s_2 t - \frac{s_4 a_2}{t} + \frac{a_2 s_3}{t^2} = a_2(-t^2 + s_1 t - s_2 - \frac{s_4}{t} + \frac{s_3}{t^2})$ $= -a_2 s_3$ on account of this relation for s_3 etc. The equation finally reduces, therefore, to

$$(1) \quad \frac{a_2^2}{a_3^2}x - \bar{x} = -a_2 s_3 + a_3 s_4 - \frac{a_4}{s_3} + \frac{a_1 a_2^2}{a_3^2} + \frac{a_2^2}{a_3} - \frac{s_1 a_2^3}{a_3^2}$$

Both sides of the equation (1) are linear, the coefficient of x on the left being the square of that on the right. Hence if x_1, x_2, x_3 are the roots of the cubic $f(x) = 0$, we must have for the roots of (1)

first on transforming the [illegible] we have

$$\frac{\partial^3}{\partial x^3}\bar{x} - X = -\frac{1}{\Delta}\frac{\partial^3 \Delta}{\partial x^3} + \frac{1}{2\Delta^2}\frac{\partial^2 \Delta}{\partial x^2}\frac{\partial \Delta}{\partial x} - \frac{1}{\Delta^3}\left(\frac{\partial \Delta}{\partial x}\right)^3 + \frac{1}{\Delta^2}\frac{\partial^2 \Delta}{\partial x^2}\frac{\partial \Delta}{\partial x} - \frac{1}{\Delta^3}\left(\frac{\partial \Delta}{\partial x}\right)^3,$$

which, on reduction [illegible]

[illegible] (1) [illegible]

The question now arises as to whether, or not, the proof of the preceding theorems by the method depending upon the use of characteristic covariants for elemental forms can be generalized. The final upon investigation that it can.

In part I., for the first of the theorems of the chain, a perpendicular was drawn from the vacuity, or rather equivalent point, of 3 lines to the 4th line of the three set of 4; for the second theorem of the chain, the perpendicular was let fall from the vacuity of the 4 lines taken 3 and 3 at a time, respectively, the vacuity of the 4 lines

with vertex $a_1 = 3, 1$.

The perpendicular bisector of the segment of these vectors is

$$\chi - (a_1 - s_1, a_2) = (\chi - a_1)\lambda$$

where λ is a [illegible]. Putting χ in simple equations for the 5 perpendicular vectors like this, for 5 lines, we have

$$\chi - (a_1 - a_2 t) + (s_1 - t)(a_2 - a_3 t) = (\chi - a_1 + a_2 t)\lambda$$

or. $\chi - a_1 + s_1 a_2 - a_1 t + a_2 t^2 - s_1 t + a_2 t^2 = \chi\lambda - a_1\lambda + a_2 t\lambda$

Now, the conjugate, for 5 lines, of $\frac{a_3}{a_2} = \frac{s_3 - a_2}{s_3 - a_3}$

$\frac{a_2}{a_3}$, that is, $\frac{s_3 - a_3}{s_3 - a_2}$ or [illegible]

multiplied by t to χ, [illegible]

$$\chi - a_1 + s_1 a_2 = \chi\lambda - a_1\lambda + a_2 t\lambda$$

and it is clear that $\chi = s$ [illegible]

[illegible] $a_1 = s_1, a_2$ [illegible]

[illegible]

$$\chi - s_1 = a_1 - a_1 \lambda$$

is an equation of the [illegible] s [illegible]

we may call this point [illegible]

the equation [illegible].

We can verify the result for [illegible] the points $i_1 - s_1 a_2$ for 4 times, say, [illegible] perpendicular is [illegible], is

$$x - \frac{s_1}{s_3}\bar{x} = a_1 - s_1 a_2 + \frac{s_1 s_4 a_3}{s_3}.$$

Symmetrizing this for 5 such lines we have

$$x - \frac{s_1 - t}{s_3 - s_2 t + s_1 t^2 - t^3}\bar{x} = a_1 - a_2 t - (s_1 - t)(a_2 - a_3 t) + \frac{(s_1 - t)(a_3 - a_4 t)s_5}{t(s_3 - s_2 t + s_1 t^2 - t^3)}.$$

If now we put for x $a_1 - s_1 a_2$ (for 5 t's) and for $\bar{x}$ $s_3 - a_4 - s_4 a_3$, and also make use of the fact that $-t_3 + s_1 t^2 - s_2 t + s_3 = \frac{s_4}{t} - \frac{s_5}{t^2}$, the result is

$$-s_5 - a_4 t^2 + s_4 a_3 t^2 = s_4 a_3 t^2 - s_5 a_4 t^2,$$

showing that the point $a_1 - s_1 a_2$ for 5 lines satisfies the equation containing the 5 perpendiculars.

If $\bar{x}$ is on the parabola $x_1 = x_1(t)$, the coefficient of t^4 for $\bar{x}$ is zero, and $\bar{x}$ is a quadratic in t.

$$x - \frac{\bar{x}}{t_6^2} = a_1 - s_1 a_2 - \frac{1}{t_6^2}(s_6 - s_1 a_6).$$

Then, for $t \to \infty$

$$t^2 x - \bar{x} = (a_1 - a_2 t) t^2 - (s_1 - t)(a_2 - a_3 t) t^2$$

$$(6\,1) \qquad -\left[\frac{(a_4 - a_5 t) s_6}{t} - (s_4 - s_3 t + s_2 t^2 - s_1 t^3 + t^4)(a_3 - a_4 t)\right]$$

From this equation we get the following relations:

$$(a_3 - a_4 t)(a_1 - a_2 t) t^2 - (s_1 - t)(a_2 - a_3 t) t^2 - \left[a_1 - s_1 a_2 + a_3 (s_1 t - t^2)\right] t^2 .$$

$$(6\,3) \quad (a_3 - a_4 t)(s_4 - s_3 t + s_2 t^2 - s_1 t^3 + t^4) - (a_3 - a_4 t)\left(\frac{s_5}{t} - \frac{s_6}{t^2}\right)$$

$$= \frac{a_3 s_5}{t} - \frac{a_3 s_6}{t^2} - s_3 a_4 + \frac{s_6 a_4}{t}$$

$$(6\,2) \quad \text{In (a)} \quad a_3 (s_1 t - t^2) t^2 = \frac{a_3 s_5}{t} - \frac{a_3 s_6}{t^2} - s_3 a_4 + \frac{s_6 a_4}{t}$$

Whence (6 1) reduces to

$$(6\,2) \qquad t^2(\bar{x} - a_1 + s_1 a_2) - (\bar{x} + a_3 s_6 - a_4 s_5) = (s_2 t^2 - s_3 t + s_4) a_2 ,$$

the linear equation of a circle, — the orthocentric circle for 6 tangent lines.

Its map-equation is

$$(6\,3) \qquad \bar{x} = a_1 - s_1 a_2 + a_3 a_4 - \frac{1}{2}\frac{a_3 s_5}{t}$$

(6 equations) ; ... of deviation ...

(65)

Now, this leaves ... terms containing t can be arranged ... ; $(+t^4 + s_1 t^3 - s_2 t^2 + s_3 t + \dots) a_4 = s_4 a_3$. Substituting

(66) $\dots - a_4 s_3 + s_4 a_3$.

Solving this equation of (66), we have

$$\frac{a_4^2}{a_3^2}\bar{x} - x = \frac{s_3 a_2 a_4^2}{a_3^2} - \frac{s_6 a_5 a_4^2}{a_3^2} \dots - a_4 s_3 + s_4 a_3$$

(67) or $\frac{a_4^2}{a_3^2} - \bar{x} = \frac{a_3^2}{a_4^2}\left(a_1 - s_1 a_2 + s_2 a_3 - s_3 a_4 + \left(s_4 a_3 - s_5 a_4 + s_6 a_3 - s_7 a_6\right)\right)$,

but (67) is (66) again. Hence (67) is the equation of the derivatives of the 1st osculants ... 1st ... Q.E.D.

Obviously, this process of elimination
can orthocentric circle for + in terms of
a determinant for 2 and 3 terms above, for
different equation, so that if we
permutation the 3rd from the end
the A" in 7 of 8 determinants in the last
left out each case, the 7 permutations
are in a circle, and the 8 circle
are in a circle, and so on indefinitely.
The number of the A" is evidently the product

$$2 = 1, 2, 3, 4, \quad \text{i.e. } 1 \cdot 2 \cdot 3 \cdot 4 = 24,$$

so that the permutations of any number,
from perpendicular from the
7 of 8 terms to the opposite
8 perpendiculars will be on a circle whose
line equation we found above. Symmetry
this line equation for 4 such circles
substitute for its parameter $\frac{dy}{dx}$, where
a's refer now to 4 t's. Thus done, the terms

containing t will cancel and the determinant for 4 lines will therefore be

In general, the equation of the nth or $2n-1$ of $2n$ lines is, a's referring to t's

$$\chi = a_1 - s_1 a_2 + s_2 a_3 - s_3 a_4 + \cdots + (-1)^n s_{n-1} a_{n-1}$$

$2n+1$ lines is $\frac{a_{n+1}}{a_n}$, where the a's refer to t $+1$. Moreover, the equation of the orthocentric circles for 4, 6, and 8 lines are respectively

(a) $\chi = a_1 - s_1 a_2 + \frac{1}{2}\frac{a_2 s_2}{t}$, a's for 4 t's

(b) $\chi = a_1 - s_1 a_2 + s_2 a_3 - \frac{1}{2}\frac{a_3 s_3}{t}$, a's for 6 t's

(c) $\chi = a_1 - s_1 a_2 + s_2 a_3 - s_3 a_4 + \frac{1}{2}\frac{a_4 s_4}{t}$, a's for 8 t's

the general equation of [illegible]

$$\chi = a_1 - s_1 a_2 + s_2 a_3 - \cdots + (-1)^{n-1} a_n + (-1)^n \frac{a_n s_n}{t}$$

Also, the general equation of [illegible]

5, 7, and 1 [illegible] :

(d) $\frac{a_2^2}{a_3^2}\bar{x} - \bar{x} = -s_3 a_2 + s_4 a_3 - s_5 a_4 + \frac{a_2^2}{a_3^2}\left(a_1 - s_1 a_2 + s_2 a_3\right)$

(e) $\frac{a_3^2}{a_4^2}\bar{x} - \bar{x} = s_4 a_3 - s_5 a_4 + s_6 a_5 - s_7 a_6 + \frac{a_3^2}{a_4^2}\left(a_1 - s_1 a_2 + s_2 a_3 - s_3 a_4\right)$

(f) $\frac{a_4^2}{a_5^2}\bar{x} - \bar{x} = -s_5 a_4 + s_6 a_5 - s_7 a_6 + s_8 a_7 - s_9 a_8 + \frac{a_4^2}{a_5^2}\left(a_1 - s_1 a_2 + s_2 a_3 - s_3 a_4 + s_4 a_5\right)$

From the general equation of the directrix for $2n+1$ paired lines is

$$\frac{a_n^2}{a_{n+1}^2}\bar{x} - \bar{x} = +(-)^{n+1} s_{n+1} a_n + (-)^{n+2} s_{n+2} a_{n+1} + (-)^{n+3} \cdots\cdots + (-)^{2n+1} s_{2n+1} a_{2n} + \frac{a_n^2}{a_{n+1}^2}\left(a_1 - s_1 a_2 + \cdots\cdots + (-)^n s_n a_{n+1}\right)$$

Section II.

[illegible] of Variation of [illegible].

[illegible] the [illegible] of [illegible],

one of a [illegible] of the central lines [illegible]

[illegible] equivalent [illegible] of [illegible] paired [illegible]

[illegible] [illegible], when the

the two lines have the equations

$$x t_1 + \frac{\bar{x}}{t_1} - 2y_1 = 0$$

and $$x t_2 + \frac{\bar{x}}{t_2} - 2y_2 = 0 ,$$

by [illegible] $x t_1 - \frac{\bar{x}}{t_1} - 2y_1 = x t_2 + \frac{\bar{x}}{t_2} - 2y_2$.

the effect of a reversal of the second kind is expressed algebraically by

$$(k t_1 + \frac{\overline{r}}{t_1} - 2r_1) + (k t_2 + \frac{\overline{r}}{t_2} - 2r_2) = 0,$$

that is, the sign before the parenthesis of the second term in the equation of the equidistant axis is changed from minus to plus. This is obvious, since now the center of the tangent circle must be found on the bisector of the angle supplementary to the angle originally bisected by the equidistant axis. Moreover, considering the proportionals as part forces, the result of a reversal of the second kind is, that from one to the equidistant axis corresponding to a reversal of the first kind, it appears that the distances will always have the same side of one of the lines, but upon opposite sides of the other; so that to obtain

The equivalent axis is perpendicular to the bisector of the direction of lines 2, the sign before $\sin\theta$ is positive when θ is measured from the bisector to the line 2 in the direction of rotation of a right screw.

Next consider 2 lines, and reverse the 2nd one. Here the intersection of the 2 equidistants is 0, giving the equidistants axis of the 3 lines, as obtained by the construction

$$x + \frac{1}{t_1} = \frac{1}{t_2} + \frac{1}{t_3}$$

$$x + \frac{1}{t_1} = \frac{1}{t_3} + \frac{1}{t_2}$$

Thus the equidistants of pairs of lines, when the 3 lines are t_1, t_2, t_3 are

$$x = \frac{2}{(t_1 - t_2)(t_1 + t_3)} + \frac{2}{(t_2 - t_1)(t_2 + t_3)} + \frac{2}{(t_3 + t_1)(t_3 + t_2)},$$

that is, the present equidistant point is the same as the equidistant point of t_1, t_2, t_3 when t_3 is changed into $-t_3$. Hence the equidistants of $t_1, t_2, -t_3$ are

$$x = \frac{2}{(t_1 - t_2)(t_1 - t_3)} + \frac{2}{(t_2 - t_1)(t_2 - t_3)} + \frac{2}{(t_3 - t_1)(t_3 - t_2)}$$

Obviously, clearing [illegible] is zero [illegible] does not [illegible]

$$x t_3 + \frac{\bar{r}}{t_3} - \bar{r}_3 \longrightarrow .$$

[illegible] of the [illegible] K_{12} [illegible] and on [illegible] three new ones, [illegible]

$$t_{12}, \quad 12 \cdot 0, \quad 12 \cdot \infty^1, \quad 12 \cdot 1$$

[illegible] lucis is reversed. [illegible] $12 \cdot \infty^1$ is obtained [illegible]

$$\left(x t_1 - 2 r_1 + \frac{\bar{r}}{t_1} \right) - \left(x t_2 - 2 r_2 + \frac{\bar{r}}{t_2} \right) = 0$$

$$\left(x t_1 - 2 r_1 + \frac{\bar{r}}{t_1} \right) + \left(x t_4 - 2 r_4 + \frac{\bar{r}}{t_4} \right) = 0 .$$

[illegible] for [illegible] points: [illegible] 4 points [illegible]:

$$X_{123} = \sum \frac{2 r_1 t_1}{(t_1 - t_2)(t_1 - t_3)}$$

$$X_{124} = \frac{2 r_1 t_1}{(t_1 - t_2)(t_1 + t_4)} + \frac{2 r_2 t_2}{(t_2 - t_1)(t_2 + t_4)} + \frac{2 r_4 t_4}{(t_4 - t_1)(t_4 + t_2)}$$

$$X_{134} = \frac{2 r_1 t_1}{(t_1 - t_3)(t_1 + t_4)} + \frac{2 r_3 t_3}{(t_3 - t_1)(t_3 + t_4)} + \frac{2 r_4 t_4}{(t_4 + t_1)(t_4 + t_3)}$$

$$X_{234} = \frac{2 r_2 t_2}{(t_2 - t_3)(t_2 + t_4)} + \frac{2 r_3 t_3}{(t_3 - t_2)(t_3 + t_4)} + \frac{2 r_4 t_4}{(t_4 + t_2)(t_4 + t_3)}$$

These expressions were obtained by [illegible], as indicated above, but it is clear that they must be found by changing the sign of t_4, &c. in the original expressions for the 4 equidistant points on a circle. In general therefore by changing the sign of t_4 and t in these formulæ for non-reversed curves, we obtain those for the n-line on the supposition that the nth line has been reversed.

Now, the equation

$$x = \frac{2t_1 t_2 (t_3 - t)}{(t_1 - t_2)(t_1 - t_3)(t_1 + t_4)} + \frac{2t_2 t_3 (t_1 - t)}{(t_2 - t_1)(t_2 - t_3)(t_2 + t_4)} + \frac{2t_3 t_1 (t_2 - t)}{(t_3 - t_1)(t_3 - t_2)(t_3 + t_4)} + \frac{2t_4 t_1 (t_2 + t)}{(t_4 + t_1)(t_4 + t_2)(t_4 + t_3)}$$

for t respectively equal to $-t_4$, t_3, t_2, t_1 becomes the equation of the points X_{123}, X_{124}, X_{134}, X_{234}. It is of the form

$$x = c - dt$$

or, writing the above coefficients α, β, γ, δ respectively, it is $x = \alpha - \beta t$,

the Lond circle for the deviated 4-line, line 4 being reversed in direction.

Using a modified form of the notation employed by Lond for the reversal of a line, let $\hat{a}_x$ denote the characteristic constants for n lines, the nth being reversed in direction. Again, it might be noted that $x = \hat{a}_1 - \hat{a}_2 t$ could be got by by changing in the original constants for a 4-line the sign of r_4 and t_4.

From (70) we get

(71) $$x - \frac{1}{t_2}\bar{x} = \frac{1}{t_2}\left(-S_4\hat{a}_3 + \frac{S_4\hat{a}_2}{t}\right) + \hat{a}_1 - \hat{a}_2 t,$$

since the conjugate of $\hat{a}_1$ for 3 things, say, is

$$\frac{\frac{2r_1}{t_1}}{\left(\frac{t_2-t_1}{t_1t_2}\right)\left(\frac{t_3+t_1}{t_3t_1}\right)} + \frac{\frac{2r_2}{t_2}}{\left(\frac{t_1-t_2}{t_1t_2}\right)\left(\frac{t_3+t_2}{t_2t_3}\right)} + \frac{\frac{2r_3}{t_3}}{\left(\frac{t_1+t_3}{t_1t_3}\right)\left(\frac{t_2+t_3}{t_2t_3}\right)},$$

or $$-\left[\frac{2r_1}{(t_1-t_2)(t_1+t_3)} + \frac{2r_2}{(t_2-t_1)(t_2+t_3)} + \frac{2r_3}{(t_1+t_3)(t_2+t_3)}\right]t_1t_2t_3,$$

that is, the conjugate of $\hat{a}_1 = -S_3\hat{a}_2$,

whereas the conjugate of a_1 is $s_3 a_{-1}$. In general, the conjugate of $\hat{a}_\alpha = (-)^{n-2} s_n \hat{a}_{n-\alpha}$, all t's of S_n being positive; however, if $\hat{s}_n$ denote that one of the t's is negative, then $\hat{a}_\alpha = (-)^{n} \hat{s}_n \hat{a}_{n-\alpha}$. Since $\hat{S}_4 = -S_4$ (71) may be written

$$t^2(x - \vec{a}_1) + (\bar{x} + \vec{s}_4 \vec{a}_3) = \hat{a}_2 t - \frac{\hat{s}_4 \bar{a}_2}{t^3}$$

(72) or $t^2(x - \vec{a}_1) - (\bar{x} + \vec{s}_4 \hat{a}_3) = (-\vec{s}_1 t^2 + \vec{s}_2 t - \frac{\vec{a}}{s_3}) \vec{a}_2$,

the line equation of the orthocentric circle for a 4-line when hand 4 is reversed.

The map equations for the 4 such circles are:

$$\begin{aligned} x &= \vec{a}_1 - \vec{s}_1 \hat{a}_2 + \frac{s_2 a_2}{2 t} \\ x &= \hat{a}_1 - \hat{s}_1 \hat{a}_2 + \frac{\hat{s}_2 \hat{a}_2}{2t} \\ x &= \hat{a}_1 - \hat{s}_1 \hat{a}_2 + \frac{\hat{s}_2 \hat{a}_2}{2t} \\ x &= \hat{a}_1 - \hat{s}_1 \hat{a}_2 + \frac{\hat{s}_2 \hat{a}_2}{2t} \end{aligned} \qquad (73)$$

while the corresponding circle for the original tetrahedron of this figure is

$$x = a_1 - s_1 a_2 + \frac{s_2 a_2}{2t}$$

Also, there are 3 possibilities for a 4-term
when two terms at a time are reversed,
Eq.,
(74)

$$x = \vec{a}_1 - \vec{s}_1 \vec{a}_2 + \frac{\vec{s}_2 \vec{a}_3}{2!}$$

$$y = \vec{a}_1 - \vec{s}_1 \vec{a}_2 + \frac{\vec{s}_2 \vec{a}_3}{2!}$$

$$z = \vec{a}_1 - \vec{s}_1 \vec{a}_2 + \frac{\vec{s}_2 \vec{a}_3}{2!}$$

Here 3 terms are taken and we have
reversing combinations of two + 3 terms
ways that give new circles. We do not need
to reverse all possible combinations of the
terms, since to reverse one but not
two (3 that remain are to reverse 2 and
to reverse two of the same are to reverse
the remaining two. What the
next relations of three terms are will
be shown in another section.

The general equation of the differential
circle for a reversed t^{th} term of a
n-term series is

$$t^{t}(X - \vec{a}_1) - (X + \vec{a}_4 \vec{a}_2) = \vec{a}_2 (-\vec{s}_1 t^2 + \vec{s}_2 t - \vec{s}_3^{\,t})$$

By summing this equation for $t_1, t_2, t_3, -t_4, t_5$ we have, calling the parameter of the curve T as usual,

$$T^2 x - \bar{x} = \hat{a}_1 T^2 - \hat{a}_2 t T^2 + \frac{(\hat{a}_3 - \hat{a}_4 t)\hat{s}_5}{t} - (\hat{a}_2 - \hat{a}_3 t)\left[(\hat{s}_1 - t)T^2 - (\hat{s}_1^2 - \hat{s}_2 t + t^2)T + (\hat{s}_3 - \hat{s}_2 t + \hat{s}_1 t^2 - t^3)\right],$$

which reduces for the t's above to the 5 orthocentric circles of a 5-line with line at reversed indication, using now the relation for 5 strings, viz.,

$$\hat{s}_3 - \hat{s}_2 + \hat{s}_1 t^2 - t^3 = \frac{\hat{s}_4}{t} - \frac{\hat{s}_5}{t^2}$$

and substituting for T the term $\frac{\hat{a}_2}{\hat{a}_3}$, we have

$$\frac{\hat{a}_2^2}{\hat{a}_3^2} x - \bar{x} = \frac{\hat{a}_1 \hat{a}_2^2}{\hat{a}_3^2} - \frac{\hat{a}_2^3}{\hat{a}_3^2} + \frac{\hat{a}_3 \hat{s}_5}{t} - \hat{a}_4 \hat{s}_5 - \frac{\hat{s}_1 \hat{a}_2^3}{\hat{a}_3^2} + \frac{\hat{a}_2^3}{\hat{a}_3^2} t + \frac{\hat{a}_2 \hat{s}_2}{\hat{a}_3^2} - \frac{\hat{a}_2^2}{\hat{a}_3}(\hat{s}_1 t - t^2) - \frac{\hat{a}_2 \hat{s}_4}{t} + \frac{\hat{a}_2 \hat{s}_5}{t^2} + \frac{\hat{a}_2}{\hat{a}_3}(\hat{s}_1 t - t^2) - \hat{a}_2 \hat{s}_2 t + (\hat{s}_1 t^2 - t^3)\hat{a}_3 + \hat{a}_3 \hat{s}_4 - \frac{\hat{a}_3 \hat{s}_5}{t}$$

This equation can be reduced, as were the equations for [illegible], and finally becomes

$$\frac{\vec{a}_1 \vec{a}_2}{\vec{a}_1 \vec{a}_2} k - \bar{k} = -\hat{s}_3 \hat{a}_2 + \hat{s}_4 \hat{a}_1 - \hat{s}_5 \hat{a}_4 + \frac{\vec{a}_1 \vec{a}_2}{\vec{a}_3 \vec{a}_4} \left(\hat{a}_1 - \hat{s}_1 \hat{a}_3 + \hat{a}_2 \hat{a}_4 \right),$$

The reciprocity of the [illegible] relation [illegible]

[illegible] of the [illegible] [illegible] [illegible]

[illegible]

$$\frac{\vec{a}_1 \vec{a}_2}{\vec{a}_3 \vec{a}_4} k - \bar{k} = -\hat{s}_2 \hat{a}_2 + \hat{s}_4 \hat{a}_3 - \hat{s}_5 \hat{a}_4 + \frac{\vec{a}_1 \vec{a}_2}{\vec{a}_3 \vec{a}_4} \left(\hat{a}_1 - \hat{s}_1 \hat{a}_3 + \hat{a}_2 \hat{a}_4 \right)$$

Thus, formulae may be found for the case of the reversal of any [illegible], and it is evident that the work may be extended to any number of [illegible].

The fact that we are supposing a [illegible] a real determinant for $\bar{k}$ [illegible] obtain [illegible] due to the generality of the angle γ in the [illegible], or the conformation. Suppose now that γ is one determinate [illegible] for the original [illegible] values for the [illegible] the [illegible] [illegible] [illegible]. The tangent of the angle γ between

these forms, is

$$\tan \varphi = -\frac{i(A-B)}{A+B}$$

where $A \equiv \frac{a_2}{a_3}$ and $B \equiv \frac{b_2}{b_3}$,

but this result, even for 5 terms, the lowest number that gives a determinant, is not simple. That is, it is not a useful result for the general case. By continuing this process of solving each pair of adjacent equations that gives a new expression, one gets to an nth,

Part III.

Section I.

Directed and Undirected Lines of the Parabola.

In the following the orthocentric circle for four lines, when these are lines of a parabola, will be derived, and the system of such circles arising on reversing one or two lines at a time will be studied. First, however, we may note some relations existing among certain circles connected with directed and undirected lines of the parabola.

The standard map equation of the parabola is

$$K = \frac{1}{(1+t)^2}, \text{ or } K = \frac{1}{(1-t)^2}$$

Replacet, as we may, in any map equation where it enters as above, by

t^2, and we have from the second form of the equation

$$x = \frac{1}{(1-t)^2}.$$

In general, the line equation of a curve may be obtained from the map equation by substituting in the following

(a) $X - \frac{\frac{dx}{dt}}{\frac{dy}{dt}} Y = f(t) - \frac{\frac{dx}{dt}}{\frac{dy}{dt}} f(t)$.

the values of the different expressions got from the map equation by the operations indicated in (a), Whence, the line equation of the parabola is

$$tx - \frac{y}{t} - \frac{t}{1-t^2} = 0$$

This is got from $tx - \frac{y}{t} - 2y = 0$, and is consequently a line directed tangent, with slope $= \frac{1}{t^2}$.

Similarly, starting with the map equation of the parabola in the form

$$x = \frac{1}{(1-t)^2},$$

the line equation is $x - \frac{y}{t} = \frac{1}{(1-t)^2}$, or

$$tx - \bar{x} = \frac{t}{1-t^2}.$$

This equation is not of form $ty + \frac{y}{t} = 0$, its chord is not t^2; it is not, therefore, the equation of a line to which a diameter may be given. It is an ordinary tangent to the parabola.

Now, t_1 and t_2 having the same meaning, from the form of the equation $tx - \frac{y}{t} = \frac{t}{1-t^2}$, we get the two following equations of the parabola

$$t_1^2 x - \bar{x} = \frac{t_1}{1-t_1^2}$$

$$t_2^2 x - \bar{x} = \frac{t_2}{1-t_2^2}.$$

Whence, by subtraction

$$(t_1^2 - t_2^2)x = \frac{t_1}{(1-t_1^2)} - \frac{t_2}{(1-t_2^2)}$$

or

$$x = \frac{1}{(1-t_1^2)(1-t_2^2)}$$

Three such tangents have an equidistant point for x as follows:

The equation

$$t_1 x - \frac{y}{t_1} = \frac{t_1}{1-t_1^2} = t_2 x - \frac{y}{t_2} = \frac{t_2}{1-t_2^2}$$

represents the equidistant axis of the two tangents with parameters t_1 and t_2.

In like manner

$$t_1 x - \frac{y}{t} - \frac{t_1}{1-t_1^2} = t_3 x - \frac{y}{t_3} - \frac{t_3}{(1-t_3^2)}$$

is another equidistant axis — that of the tangents represented by the two members of this equation. Simplifying each of these equations, we have

$$t_1 t_2 x + y = \frac{t_1 t_2 (1 + t_1 t_2)}{(1-t_1^2)(1-t_2^2)}$$

and

$$t_1 t_3 x + y = \frac{t_1 t_3 (1 + t_1 t_3)}{(1-t_1^2)(1-t_3^2)}$$

These axes intersect in the point given by

$$(t_2 - t_3) x = \frac{t_2 (1 + t_1 t_2)}{(1-t_1^2)(1-t_2^2)} - \frac{t_3 (1 + t_1 t_3)}{(1-t_1^2)(1-t_3^2)}$$

or

$$x = \frac{1 + t_1 t_2 + t_1 t_3 + t_2 t_3}{(1-t_1^2)(1-t_2^2)(1-t_3^2)} = \frac{1 + S_2}{\Pi (1-t_1^2)}$$

which is the equidistant point of 3 directed lines of the parabola with elements $1/t_1^2$, $1/t_2^2$, $1/t_3^2$.

Consider next two lines of the parabola when the line is of the form

$$tx - y = \frac{t}{1-t^2}.$$

This is the undirected line. Let the parameters be t_1 and t_2, which naturally suggest themselves in connection with the parameters t_1' and t_2' just considered. The lines are

$$t_1 x - \bar{x} = \frac{t_1}{1-t_1}$$

and

$$t_2 x - \bar{x} = \frac{t_2}{1-t_2}$$

and they intersect in the point

$$x = \frac{1}{(1-t_1)(1-t_2)},$$

For their second intersection, we have

$$x = \frac{1-t}{(1-t_1)(1-t_2)(1-t_3)},$$

the result on the 3 intersections of the lines with $\bar{x}$, $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$. The parameter is

$$x = \frac{1}{(1-t_1)(1-t_2)(1-t_3)}.$$

Now, the parameters $-t_1$ and $-t_2$ are also suggested by the parameters of the directed lines just given. That is, the parameters t_1' and t_2' suggest not only t_1 and t_2 for undirected lines, but also $-t_1$ and $-t_2$. Similarly, for

adding these we find -1, and by cross-multiplying by z, we have

$$z_1 + z_2 = \quad 1 + z_1 t + z_1 z t = z_3 .$$

Hence we see generally that if z_1 and z_2 are the coordinates of the vertices of a triangle inscribed in the circle $|z| = 1$, the coordinates of the vertex z_3 of the triangle formed by the tangents at the vertices z_1 and z_2 are the coordinates of the point whose abscissa and ordinate are respectively equal to $\frac{1}{t_1}$, $\frac{1}{t_2}$ and $-\frac{1}{t_1}$, $-\frac{1}{t_2}$, $-\frac{1}{t}$ in the inscribed triangle of the circle whose abscissa and ordinate are equal to $\frac{1}{t_1}$, $\frac{1}{t_2}$, and $\frac{1}{t_3}$.

Next take the equidistant point ζ of the inscribed triangle when one of the vertices is removed to infinity, as z_3, the circle will circumscribe $\triangle z_1 z_2$. In this case the equidistant equal of the two lines given by $z_1 z$ and $z_2 z$ is $t_1 t_2 z + \zeta = \frac{t_1 t_2 (1 + \zeta t_1 t_2)}{(1 - t_1^2)(1 - t_2^2)}$. But

for $t_1 x - \frac{\bar{x}}{t_1} - \frac{\bar{y}_1}{1-t_1^2} = 0$ and $t_3 x - \frac{\bar{x}}{t_3} - \frac{t_3}{1-t_3^2} = 0$,
when the latter is reversed in direction,
is given by

$$t_1 x - \frac{\bar{x}}{t_1} - \frac{\bar{y}_1}{1-t_1^2} = t_3 x - \frac{\bar{x}}{t_3} - \frac{t_3}{1-t_3^2}$$

or

$$t_1 t_3 x - \bar{x} = \frac{t_1 - t_3 t_1 \bar{y}_1 - t_3}{(1-t_1^2)(1-t_3^2)}$$

Hence substituted in the place of $\bar{x}$

$$x = \frac{t_1 + t_3 - t_1 t_3 \bar{y}_1 - t_3}{(1-t_1^2)(1-t_3^2)}$$

Let us now find the value of y for the intersection of the [illegible] formed by [illegible] with t_2 [illegible] t_1, t_2, $-t_1$ and $-t_2$, $-t_3$, t_3. We have the equations

$$t_1 x - \bar{x} = \frac{t_1}{1-t_1^2}$$

$$t_3 x + \bar{x} = \frac{t_3}{1+t_3^2},$$

and thence we have

$$\bar{x} = \frac{1}{(1-t_1^2)(1+t_3^2)}$$

Eliminating from these two the $\bar{x}$ coordinate,
there results

$$x = \frac{1 - t_1 t_3 \bar{y}_1}{(1-t_1^2)(1+t_3^2)},$$

the parameter-values of the 2 remaining lines now under consideration will be

$$\dots$$

Further, the tangents from ∞ to the circle meet it in the points $-t_1$ and $-t_2$ intersect in the common point of

$$t_1 x + \bar{x} = \frac{t_1}{1+t_1}$$

$$t_2 x + \bar{x} = \frac{t_2}{1+t_2},$$

that is, in $x = \frac{1}{(1+t_1)(1+t_2)}$,

which for a line x with parameters $-t_1$, $-t_2$ and t_3 becomes $\dots$.

Thus if, of course, the parameter-values of the lines $-t_1, -t_2$, and t_3, the condition

$$\Delta_1(x_0 - t_1, -t_2, t_3) = \frac{1}{(1+t_1)(1+t_2)(1-t_3)}$$

vanishing now the strokes for $\Delta(\eta\ t_1, t_2, -t_3)$ and $\Delta(\eta\ -t_1, -t_2, t_3)$ $\dots$

$$\dots$$

$$\dots$$

point of the 3 directed lines given by slopes $\frac{1}{t_1^2}$, $\frac{1}{t_2^2}$, $\frac{1}{t_3^2}$ with the 3rd line reversed in direction. Hence the mid-point of the circumcenters of the triangles formed by the sets of directed lines of the parabola, with slopes $\frac{1}{t_1}, \frac{1}{t_2}, -\frac{1}{t_3}$ and $-\frac{1}{t_1}, -\frac{1}{t_2}, \frac{1}{t_3}$ is just that of the equidistant point of the three directed lines with slopes $\frac{1}{t_1^2}, \frac{1}{t_2^2}, \frac{1}{t_3^2}$, the last line being reversed in direction.

Similar theorems are true for the mid-point of the circumcenters of the three lines given by $t_1, -t_2, t_3$ and $-t_1, t_2, -t_3$, and the equidistant point of the directed lines given by t_1^2, t_2^2, t_3^2 with the second line reversed in direction. And similarly for the three lines given by $-t_1, t_2, t_3$ and $t_1, -t_2, -t_3$, and the directed lines with parameters t_1^2, t_2^2, t_3^2 with the first line reversed. Thus the four mid-points are (a) $X_1 = \frac{1 + t_1t_2 + t_1t_3 - t_2t_3}{(1-t_1^2)(1-t_2^2)(1-t_3^2)}$

$$(b)\ k_2 = \frac{1 + t_1 t_2 - t_1 t_3 - t_2 t_3}{(1-t_1^2)(1-t_2^2)(1-t_3^2)}$$

$$(c)\ k_3 = \frac{1 - t_1 t_2 + t_1 t_3 - t_2 t_3}{(1-t_1^2)(1-t_2^2)(1-t_3^2)}$$

$$(d)\ k_4 = \frac{1 - t_1 t_2 - t_1 t_3 + t_2 t_3}{(1-t_1^2)(1-t_2^2)(1-t_3^2)}$$

The reciprocal of the common denominator is

$$\frac{1}{k_{1234}} = (1-t_1^2)(1-t_2^2)(1-t_3^2)$$

The ratio of corresponding sides of the quadrilaterals

$$\frac{\tau_1}{\tau_1'} = \frac{t_1}{1-t_1^2}$$

$$\frac{\tau_2}{\tau_2'} = \frac{t_2}{1-t_2^2}$$

is $$\chi = \frac{1}{(1-t_1^2)(1-t_2^2)}$$

and for three such quadrilaterals, as (a), (b), (c),

$$\chi = \frac{1 - t^2}{(1-t_1^2)(1-t_2^2)(1-t_3^2)},$$

the circumcircle of the three directed tangents will be

$$\chi = (1-t^2)(1-t_1^2)(1-t_2^2)(1-t_3^2).$$

which is the condition for the four circumcircles of the four mid-points (a), (b), (c), (d) just given to coincide.

Hence, remembering that with these directed lines of the parabola may be associated

four pairs of triangles of corresponding lines, viz.
these pairs by the formulæ

(i) t_1, t_2, t_3 and $-t_1, -t_2, -t_3$

(ii) $t_1, t_2, -t_3$ " $-t_1, -t_2, t_3$

(iii) $t_1, -t_2, t_3$ " $-t_1, t_2, -t_3$

(iv) $-t_1, t_2, t_3$ " $t_1, -t_2, -t_3$

We may show that the centroids of the four mid points, each referring to a pair of corresponding triangles in general form a parallelogram whose centre is the centroid of the triangle of directed lines with which the corresponding triangles are associated. We have thus associated with each mid-point a parallelogram and with each of the parallelograms a point. Consider, then, the [illegible] of the mid-points of the [illegible] triangles of the [illegible] triangle is the circumcentre of that triangle.

Consider next a directed lines of a pa-

Parabola with elements, say, $\frac{1}{t_1}$, $\frac{1}{t_2}$, $\frac{1}{t_3}$, $\frac{1}{t_4}$. Associated with these 4 lines are the 8 undirected lines with elements $\frac{1}{t_1}$, $\frac{1}{t_2}$, $\frac{1}{t_3}$, $\frac{1}{t_4}$ and their negatives. Also, there may be combinations of these undirected lines. In the case of 4 lines there are 4 sets of three lines. Each of the four sets has a circumcircle whose center is the centroid of four corresponding mid-points of 4 pairs of circumcenters of triangles of undirected tangents, but the circumcenters of the 4 circles of 3 of 4 lines are on a circle. Therefore the four centroids of mid-points as above described are on a circle.

Again, it is easily seen that in the case of the 4 directed 3-lines of a directed four-line, there are at least 8 undirected lines, 16 mid-points in sets of 4, and 32 circumcircles in sets of 8 with their

sets of 8 circumcenters.

It has just been shown that a directed 3-line with its six associated undirected lines gave a centroid which is the center of the circumcircle of the directed 3-line. For a directed 4-line, the 4 centroids are on a circle, the center circle of the 4-line. It was also shown that, given 3 directed lines, the mid-point of the circumcenters of the two triangles of undirected lines with parameters t_1, t_2, t_3 and $-t_1, -t_2, t_3$ is the in-center of the triangle of associated directed lines. If we take 4 directed lines, the 4 incenters, or, better, equidistant points, are on a Lemoine circle. Hence the 4 mid-points, one each for every 2 triangles of associated undirected lines are on a circle, the Lemoine circle of the four lines. Thus, the mid-point circle for a 4-line is the Lemoine

circle for those lines, while the centroid circles is the center circle of the same.

Again, the center of the circumcircle of the undirected lines t_1, t_2, t_3, t_4 is

$$C' = \frac{1}{(1-t_1)(1-t_2)(1-t_3)(1-t_4)};$$

that of $-t_1, -t_2, -t_3, -t_4$ is

$$C'' = \frac{1}{(1+t_1)(1+t_2)(1+t_3)(1+t_4)}.$$

The product of these two strokes is

$$C'C'' = \frac{1}{(1-t_1^2)(1-t_2^2)(1-t_3^2)(1-t_4^2)},$$

which is the center of the center circle for 4 directed lines given by $t_1^2, t_2^2, t_3^2, t_4^2$.

The equation

$$C'C'' = \frac{1}{(1-t_1^2)(1-t_2^2)(1-t_3^2)(1-t_4^2)}$$

defines, as is well-known, a complex involution of the plane. Consider the equations of the axes whose endpoints are pairs of points of this involution. These points are for the undirected lines t_1, t_2, t_3, t_4 and $-t_1, -t_2, -t_3, -t_4$;

there are, therefore, at least four such points, each for its being the center of the circle and of the inscribed circle. The two circles given by t_1, t_2, t_3, t_4 and their negatives are

$$X = \frac{1-t}{(1-t_1)(1-t_2)(1-t_3)(1-t_4)} \equiv \frac{1-t}{A}$$

and $$\chi = \frac{1+t}{(1+t_1)(1+t_2)(1+t_3)(1+t_4)} \equiv \frac{1+t}{B},$$

Taking conjugates of $Ax - 1 = -t$, we have

$$\frac{A\bar{\chi}}{S_4} - 1 = -\frac{1}{t}.$$

Multiplying these equations, we result in

$$A\chi\bar{\chi} - \bar{\chi} - \chi = 0$$

Similarly, $$B\chi\bar{\chi} - \bar{\chi} - \chi = 0.$$

These are the equations in conjugate coordinates of the center circles of the two sets of inscribed tangent circles. Combining these equations, we have

$$AS_4\chi^2 - S_4\chi - S_4\chi(BS_4\chi - 1) = 0$$

or $$S_4\chi(A - B)\chi = 0,$$

which shows that these two circles intersect in imaginary points and that

, $\bar{z}=0$, at the origin. It is evident from the map equations of those circles that they are in the region for $t=1$ and $t=-1$. Moreover, their centers and the origin lie on a line for

$$\begin{vmatrix} \frac{1}{A} & \frac{S_4}{A} & 1 \\ \frac{1}{B} & \frac{S_4}{B} & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

Hence, the two center circles of associate reciprocal bicircular quartics where for t unity are t_i $(i=1,3,3,4)$ and their negatives are in the origin and have contact at the origin. Previous work shows that the same thing is true for the seven additional pairs of center circles that may be selected from the two sets of four t's each. A pair of circles would be given, for instance, by $t_1, t_4, t_3, -t_4$ and $-t_1, -t_4, -t_3, t_4$.

The circles: $A_4\bar{z}-z-S_4\bar{z}=0$ for t_i, $i=1,3,4$

and $B_4\bar{z}-z-S_4\bar{z}=0$ for $-t_i$, $i=1,3,4$

have as their common tangent at the origin

$$S_4 \chi + \bar{\chi} = 0 .$$

The circles: $A_{S_4} \chi\bar{\chi} - \bar{\chi} - S_4 \chi = 0$ for $t_1, t_2, -t_3, -t_4$

and $\ldots \bar{\chi} - \bar{\chi} \ldots \chi = 0$ for $-t_1, -t_2, t_3, t_4$

have as their common tangent at the origin

the same line $S_4 \chi + \bar{\chi} = 0$

But the circles: $A_4 \chi\bar{\chi} - \bar{\chi} + S_4 \chi = 0$ for $t_1, t_2, t_3, -t_4$

and $B_4 \chi\bar{\chi} - \bar{\chi} + 4\chi = 0$ for $-t_1, -t_2, -t_3, t_4$

have $S_4 \chi - \bar{\chi} = 0$.

Thus, of the eight pairs of circles like the ones first considered four pairs have as their common tangent at the origin

$$S_4 \chi + \chi = 0 ,$$

while the remaining four pairs have

$$S_4 \chi - \bar{\chi} = 0 .$$

Thus, of these 16 circles 8 belong to one parabolic system and 8 to another, and every circle of one system intersects every circle of the other system orthogonally.

Section 2.

Relations of the Eight Bound Circles for Four Directed Lines.

Consider the circles on the equidistant points of four directed lines. Let the lines be those of a parabola. The equations of the lines will then be of the form, already shown, viz.

$$tx - y/t = \frac{t}{1-t^2}.$$

When the lines with parameters t_1, t_2, t_3, t_4 are reversed in all possible ways to give new circles, there are the following cases. Call the lines 1, 2, 3, 4. Reverse in succession each of the four lines and obtain four new circles. Reverse lines 1 and 2, 1 and 3, and 1 and 4 and get three new circles. Now, the effect of reversing lines 1 and 2 is the same as that of reversing lines 3 and 4, while

reversing three times at a time is the same as reversing the remaining one of the four. Thus, counting the circle for the required reflections of the lemma, there are eight circles in all.

Lord has shown in his "Geometry of Metric Theorems Concerning Circles in a Plane", already referred to, that these eight circles fall into two sets of four, each circle of one set is orthogonal to every circle of the other. Now, in order to have 4 circles on two points and 4 circles about these points, it requires four constants to fix the two points, four more for the circles on the 4 points, and still four more for the 4 circles about the points, making 12 constants in all, but the two orthogonal sets of circles each

* Trans. American Math. Society, Vol. 1, No. 3, pp. 323-338

determined uniquely by four directed lines for which [illegible], it is, therefore, evident that these sets of axes are subject to relations.

The equidistant point of the lines of a parabola with elements $\frac{1}{t_1}$, $\frac{1}{t_2}$, $\frac{1}{t_3}$ was

$$x = \frac{1 + t_1t_2 + t_1t_3 + t_2t_3}{(1-t_1^2)(1-t_2^2)(1-t_3^2)}.$$

This point for four lines is

$$x = \frac{(1 + S_2 - S_1t + t^2)(1-t^2)}{(1-t_1^2)(1-t_2^2)(1-t_3^2)(1-t_4^2)}$$

or

$$x = \frac{1 + S_2 + S_4 - (S_1+S_3)t}{(1-t_1^2)(1-t_2^2)(1-t_3^2)(1-t_4^2)}$$

Write this as $x = \frac{(1 + S_2 + S_4) - (S_1+S_3)t}{P}$

in which $-t^4 + S_1t^3 - S_2t^2 = -S_3t + S_4$ from the relation for four things, and $P \equiv (1-t_1^2)(1-t_2^2)(1-t_3^2)(1-t_4^2)$. The above is the equation of the Four Circle on the four equidistant points of four directed lines of a parabola.

We may note in passing that the equidistant point for 3 lines was $x = \frac{1+S_2}{P}$

, S and P referring to three t's. If we now reverse lines 1, 2, 3 in succession, we change S_2 but do not change P in value. The equation

$$x = \frac{1 + S_2 - 2S_1 t + t^2}{P},$$

which becomes $x = \frac{1 - S_2 + \frac{2S_3}{t}}{P}$,

for $t = -t_1 - t_2 - t_3$ picks up the three exterior centers of the 3-line. It is the circle on these three points. It is to be observed here, too, that if in the expression for the center of the bond circle for four lines just formed, viz., $x = \frac{1 + S_2 + S_4}{P}$,

S_2 is taken as the parameter, the equation is that of a straight line so long as only one t is changed in sign, or again when two t's are changed in sign. S_2 can be used as a parameter since $\frac{S_2}{S_4} = \frac{S_2}{S_4}$, for four t's, and it therefore moves on a line just as x in $x = t$ moves

on a circle. Whence we conclude that, from this point of view, four centers of the eight circles lie on one line and four on another line. Take

$$P_X = 1 + S_2 + S_4 .$$

The conjugate is $\frac{P}{S_4}\bar{X} = 1 + S_2 + S_4$

Whence, $\quad S_4 X - \bar{X} = \Omega ,$

the equation of the line on conjugate coördinates on which is the center of the original family of circles.

We may also write in passing that as the equidistant point of the 3-line is

$$k = \frac{1+\ldots}{P}$$

we may put $\quad 2P_X = (1-t_1)(1+t_3)(1+t_5) + (1+t_1)(1+t_3)(1+t_5)$

or $\quad 2P_X = \pi^3(1-t_1) + \pi^3(1+t_1)$

The conjugate is: $\quad 2P\bar{X} = -\pi^3(1-t_1)S_3 + \pi^3(1+t_1)S_3$

$$\therefore P(S_3 X - \bar{X}) = \pi^3(1+t_1)S_3$$

$$P(S_3 X + \bar{X}) = \pi^3(1-t_1)S_3$$

$$\therefore X^2 - \bar{X}^2 = \frac{S_3^2 \pi^6}{P^2} .$$

The last equation is the rectangular hyperbola on the four equidistant points of a 3-line. Different equidistant points are obtained, if among the 4 parameters the signs are changed.

Let us now return to four directed lines of the parabola and let us suppose the fourth line is reversed in direction. The equidistant axis for lines with parameters t_1' and t_2' was

$$t_1 t_2 x + \bar{x} = \frac{(1+t_1 t_2) t_1 t_2}{(1-t_1^2)(1-t_2^2)}.$$

That for lines t_1' and t_4', when t_4' is reversed in direction is

$$t_1 t_4 x - \bar{x} = \frac{(1-t_1 t_4) t_1 t_4}{(1-t_1^2)(1-t_4^2)}$$

Eliminating $\bar{x}$ from these two equations, we obtain

$$x = \frac{1 + t_1 t_2 - t_1 t_4 - t_2 t_4}{(1-t_1^2)(1-t_2^2)(1-t_4^2)}.$$

Symmetrizing for four t', there results

$$x = \frac{1 + \Sigma_2 - \Sigma_3 t + t^2}{(1-t_1^2)(1-t_2^2)(1-t_3^2)(1-t_4^2)}$$

or $$\kappa = \frac{1 + {}^{4}S_2 + {}^{4}S_4 - ({}^{4}S_1 + {}^{4}S_3)t}{p},$$

the equation of the Ford circle for four lines with line four reversed, ${}^{4}S_i$ designates the symmetric functions of the 4 t's when line 4 is reversed. In like manner

$$\kappa = \frac{1 + {}^{12}S_2 + {}^{12}S_4 - ({}^{12}S_1 + {}^{12}S_3)t}{p}$$

is the equation of the Ford circle for four lines when lines 1 and 2 are simultaneously reversed in direction. Denote the above two circles briefly by $+R$ and $12R$, and use similar designations for the circles obtained on reversing other lines. Let us obtain the equations of the two lines of centers of the two sets of orthogonal Ford circles for the four lines under consideration, and first that of the line on the two points on which the hyperbolic set of four cir-

ales meet. The centers of the elliptic net of four are on this locus, and in particular the center of the circle for the regular tetrahedron of this kind, the center of P, say, in case of 12 R. The centers are respectively

$$x = \frac{1+S_2+S_4}{P} \quad \left.\right\} \text{(S for 4 t's.)}$$
$$\bar{x} = \frac{1+\bar{S}_2+\bar{S}_4}{P} \quad \left.\right\}$$

The locus in point coords is

$$\begin{vmatrix} 1 & x & \bar{x} \\ 1 & \frac{1+S_2+S_4}{P} & \frac{S_4(1+S_2+S_4)}{P} \\ 1 & \frac{1+\bar{S}_2+\bar{S}_4}{P} & \frac{S_4(1+\bar{S}_2+\bar{S}_4)}{P} \end{vmatrix} = 0$$

Since $\bar{S}_4 = S_4$, the equation of the line of centers of the elliptic circles, i.e. 0, 12 R, 13 R, and 14 R is

$$S_4 x - \bar{x} = 0,$$

and it is to be verified that a certain necessary power of this matrix is $\frac{1+\ldots+S_4}{P}$ times the ...

Again similarly, the basic equation of the hyperbolic set of 4 circles, viz., 1R, 2R, 3R, and 4R is

$$S_4 \chi - \overline{\chi} = 0.$$

But $S_4 = S_4^1 = S_4^2 = S_4^3 = S_4^4 = -t_1 t_2 t_3 t_4$. When the above equation is

$$S_4 \chi + \overline{\chi} = 0,$$

which from the former [illegible] fundamental to $S_4 \chi - \overline{\chi} = 0$, [illegible].

Now, the Final circle was

$$\chi = \frac{1 + S_2 + S_4 - (S_1 + S_3) t}{P}.$$

Then $$\overline{\chi} = \frac{S_4(1 + S_2 + S_4) - S_4(S_1 + S_3) 1/t}{P}.$$

For shortness let $1 + S_2 + S_4 = a$ and $S_1 + S_3 = b$. These equations become with the abbreviations $\chi = \frac{a - bt}{P}$

and $\overline{\chi} = \frac{S_4 (a - \frac{b}{t})}{P}$

from which we get

$$P^2 \chi \overline{\chi} - aP(S_4 \chi + \overline{\chi}) = S_4(b^2 - a^2),$$

the equation of the inquiré circle in conjugate coordinates. In like manner

$$P^2 x\bar{x} - a P(\delta_4 x + \bar{x}) = \tfrac{1}{4}(b^2 - a^2)$$

is the circle Ik in the same coordinates. Now, where does the line of centers $\delta_4 x - \bar{x} = 0$ intersect the circle for the inquirial deviation of the line — the circle k? In the purpose of ascertaining the points of intersection, combine

$$P^2 x\bar{x} - aP(\delta_4 x + x) = \tfrac{1}{4}(b^2 - a^2)$$

and $\delta_4 x - \bar{x} = 0$,

The result is $x^2 - \frac{2a}{P} x = \frac{b^2 - a^2}{P^2}$

or $x = \frac{a \pm b}{P}$,

the two points of intersection. In like manner $\delta_4' x - \bar{x} = 0$, or $\delta_4 x + x = 0$ intersects the circle Ik in the points

$$x = \frac{a' \pm b'}{P}$$

between the two the line on the points $\frac{a+b}{P}$ and $\frac{a'+b'}{P}$. It is

$$\begin{vmatrix} 1 & x & \bar{x} \\ 1 & \frac{m}{p} & \frac{-m}{p} \\ 1 & \frac{n}{p} & \frac{n}{p} \end{vmatrix} = 0$$

in which $m = a + b$ and $n = a + b'$. Noting that $\delta_4 = -4$, this equation reduces to

$$x - \frac{m-n}{\delta_4(m+n)}\bar{x} = \frac{2mn}{p(m+n)}.$$

The element of this being

$$\frac{m-n}{\delta_4(m+n)}, \text{ or better } \frac{m-n}{\delta_4(m+n)}$$

since n may refer to the / changed in deviation. Let m refer, hereafter, to circles subtended by the two points lying of one side of x, the zero point, and n to those on the other side. Then, m has three positions only, due to the circles B $1B$, $2B$, $4B$, and n to $1R$, $2R$, $3R$, and $4R$. Because $\frac{\Sigma x}{p} = 0$, the limit on the points $x =$

is $K = \frac{m'' - n''}{S_4(m'' + n'')} \quad \bar{K} = \frac{2 m'' n''}{m'' + n''}$

with obvious notation $\frac{m'' - n''}{S_4(m'' + n'')}$.

Now, $m - n' = 1 + S_1 + S_2 + S_3 + S_4 - 1 - S_1' - S_2' - S_3' - S_4'$

$= 2t_1(t_2 + t_3 + t_4) + 2t_1(t_2 t_3 t_4) + 2t_1 + 2t_1(t_2 t_3 + t_2 t_4 + t_3 t_4)$

$= 2t_1(1 + S_1 + S_2 + S_3)$

with obvious notation for t_2, t_3, t_4.

$m + n' = 1 + S_1 + S_2 + S_3 + S_4 + 1 + S_1' + S_2' + S_3' + S_4'$

$= 2 + 2(t_2 t_3 + t_2 t_4 + t_3 t_4) + 2(t_2 + t_3 + t_4) + 2(t_2 t_3 t_4)$

$= 2(1 + S_1 + S_2 + S_3)$

with obvious notation for t_2, t_3, t_4.

Consequently, the ratio

$\frac{m - n'}{S_4(m + n')} = \frac{2t_1(1 + S_1 + S_2 + S_3)}{2S_4(1 + S_1 + S_2 + S_3)} = \frac{1}{t_2 t_3 t_4}$,

that is, the ratio of the harmonic means of a, b and $\frac{1}{a}, \frac{1}{b}$ is

$\frac{1}{t_2 t_3 t_4}$.

In this manner we may find the ratio that the harmonic mean of three terms

the points $\frac{14}{u} + \frac{17}{p}$ and $\frac{4}{u} + \frac{7}{p}$

which was $\frac{14}{S_4(14t + u)}$

we reduced to

$$-\frac{-t_1(1+t_1+t_1t_2+t_1t_2t_3)}{2S_4(1+S_1+S_2+S_3)} = -\frac{1}{t_2t_3t_4}$$

in which the S's are for t_2, t_3, $-t_4$.

But it is seen that these elements are ... in ... . However this division is what the ... exponentially ... and ... .

Further, it can be shown that the ... if ... would be [illegible]

$$\frac{2}{p} \qquad \frac{2}{p}$$

$$\frac{2S_1(1+t_1+t_1t_2+t_1t_2t_3)}{2S_4(1+S_1+S_2+S_3)} = \frac{1}{t_2t_3t_4},$$

that is, this ... formula is proportional to the ... and ... . this ... it is seen that ... the lines joining the various points of intersection of the ... of ... of ...

l_{∞} is cut by the lines $x+\lambda y \pm \bar{x} = 0$. In this way we get four lines, viz., those joining the following pairs of points, where a, b, and P retain the same significations as before:—

(α)	$\frac{a+b}{P}$	and	$\frac{a'+b'}{P'}$
(β)	$\frac{a+b}{P}$	"	$\frac{a'-b'}{P'}$
(γ)	$\frac{a-b}{P}$	"	$\frac{a'-b'}{P'}$
(δ)	$\frac{a-b}{P}$	"	$\frac{a'+b'}{P'}$.

The elements of these lines are respectively

(α)	$\frac{1}{t_2 t_3 t_4}$	
(β)	$\frac{\pm 2i}{S_4(1+S_4)}$	S_1, S_2, S_3 for t_2, t_3, t_4,
(γ)	$-\frac{1}{t_2 t_3 t_4}$	
(δ)	$-\frac{S_1+S_3}{S_4(1+S_4)}$	$(1,4), (2,3), (3,4)$,

It will thus appear that the elements of the lines on the pairs of points (α) and (γ) and (β) and (δ) are negatives and that the lines are perpendicular. These four lines of course form a circumscribed quadrangle.

In a similar manner each of the circles of the elliptic set can be taken in its intersections with its lines of conics joined to the similar intersections of the circles of the hyperbolic set with their lines of conics. In all there are 64 lines, and these are all the lines possible that can be drawn in the manner described. These 64 lines are divided into groups and perpendicular pairs of lines. The groups also have their perpendicular groups. Consider first the elements of the 64 lines obtained by joining the intersections in all possible ways, as described above. The general form of every element is

$$\frac{m-n}{4(m+n)},$$

where m is equal to $1+\delta_2+\delta_4+(\delta_1+\delta_3)$ and refers to the elliptic set of circles. On the other hand n is equal to $1+\delta_2+\delta_4+(\delta_1+\delta_3)$ and

refers to the hyperbolic set of circles. Substituting for these symmetric functions their values in the t's we get all possible chinants. Further, $m = a \pm b$ and $n = a \pm b$ but of course refer to different sets of circles. It will be best to use a and b with the proper superscript to indicate the points. In every case below $S_4 = t_1 t_2 t_3 t_4$. $\bar{a}$ etc means a for the circle obtained on reversing limit 1 etc. The chinants of the lines follow in sets. The sets are those obtained by joining the intersections of a pair of circles like O and $1R$ with the lines of circles $\pm 4 \; x \pm \bar{x}$ etc. In the middle of the page is given the set of four lines, on the left of the page the pairs of points joined, and on the right the [illegible] T.

Set of Four Lines for O and $4R$

Lines joining.	Chinants

$\frac{a+b}{P}$	$\frac{a+b}{P}$	$\frac{1}{t_1 t_2 t_3}$
$\frac{a+b}{P}$ "	$\frac{a-b}{P}$	$\frac{s_1+s_3}{s_4(1+s_2)}$ (s for t_1, t_2, t_3)
$\frac{a-b}{P}$ "	$\frac{a-b}{P}$	$-\frac{1}{t_1 t_2 t_3}$
$\frac{a-b}{P}$ "	$\frac{a+b}{P}$	$-\frac{s_1+s_3}{s_4(1+s_2)}$ (s for t_1, t_2, t)

4 Lines for 12R and 3R

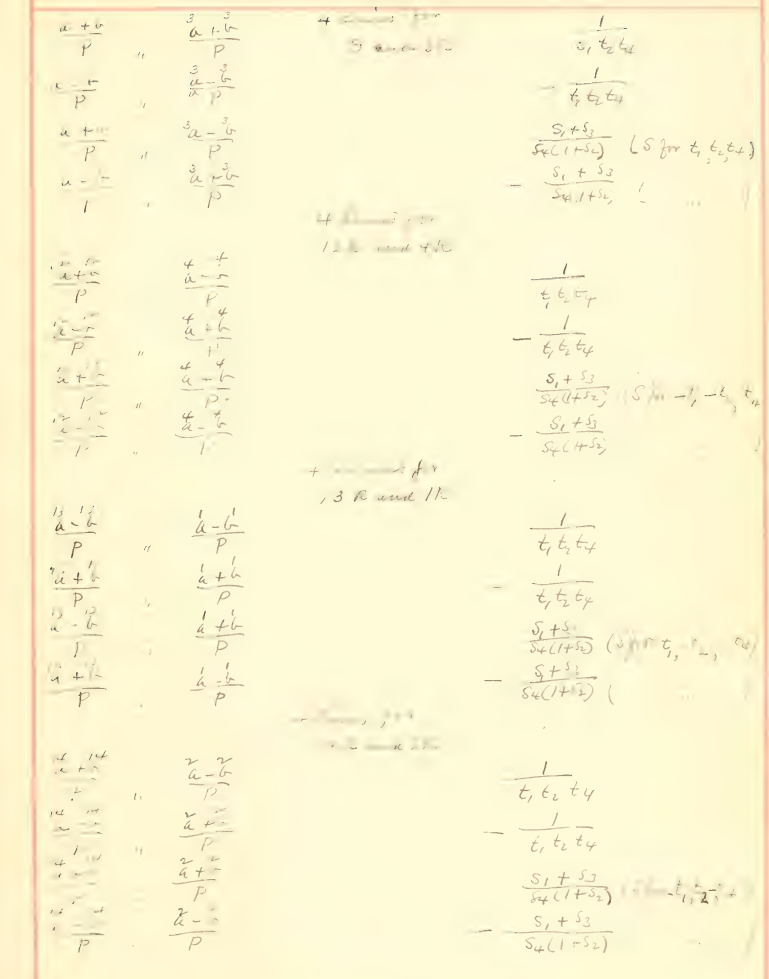
$\frac{12a+12b}{P}$ "	$\frac{3a-3b}{P}$	$\frac{1}{t_1 t_2 t_3}$
$\frac{12a-12b}{P}$ "	$\frac{3a+3b}{P}$	$-\frac{1}{t_1 t_2 t_3}$
$\frac{12a+12b}{P}$ "	$\frac{3a+3b}{P}$	$\frac{s_1+s_3}{s_4(1+s_2)}$ (s for t_1, t_2, t)
$\frac{12a-12b}{P}$ "	$\frac{3a-3b}{P}$	$-\frac{s_1+s_3}{s_4(1+s_2)}$ (")

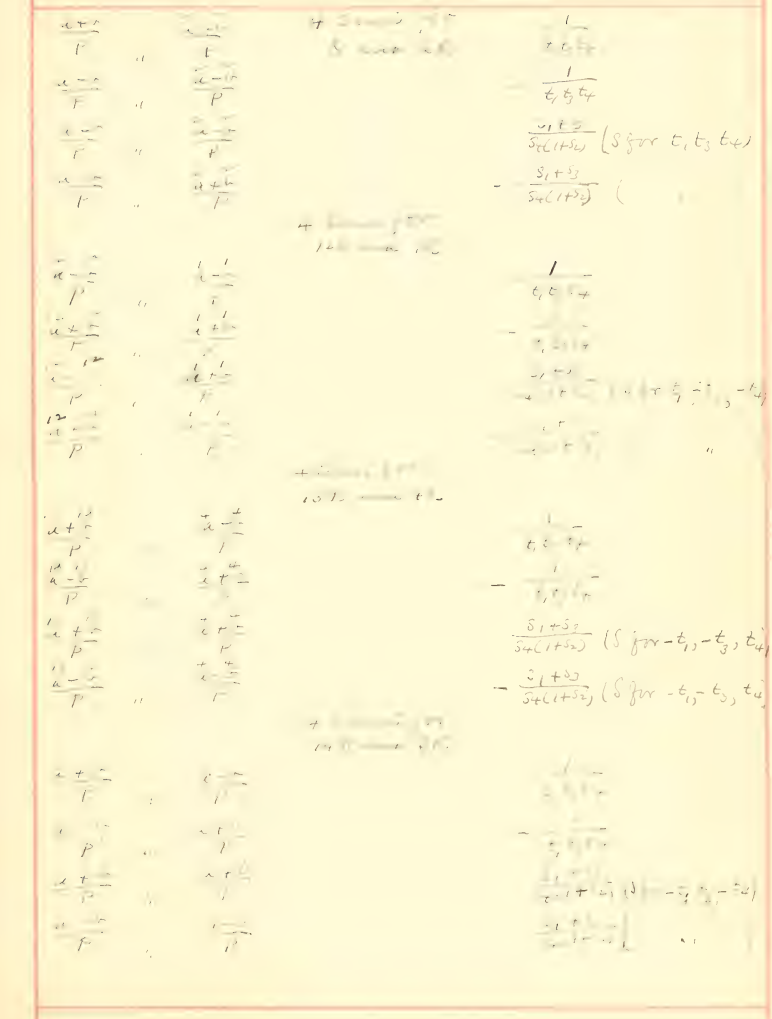
4 Lines for 13R and 2R

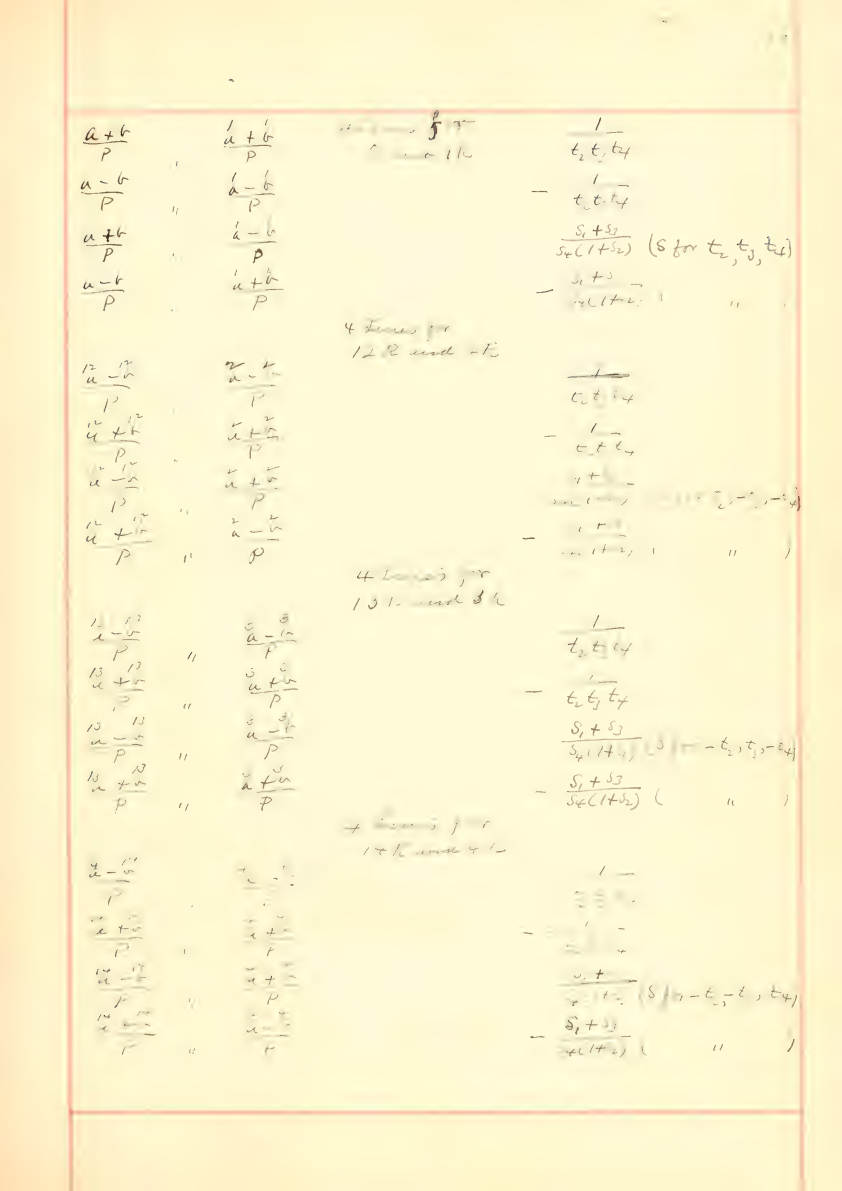
$\frac{13a+13b}{P}$ "	$\frac{2a-2b}{P}$	$-\frac{1}{t_1 t_2 t_3}$
$\frac{13a-13b}{P}$ "	$\frac{2a+2b}{P}$	$-\frac{1}{t_1 t_2 t_3}$
$\frac{13a+13b}{P}$ "	$\frac{2a+2b}{P}$	$-\frac{s_1+s_3}{s_4(1+s_2)}$ (s for t_1, t_2, t)
$\frac{13a-13b}{P}$ "	$\frac{2a-2b}{P}$	$-\frac{s_1+s_3}{s_4(1+s_2)}$ (")

4 Lines for 14R and 1

$\frac{14a-14b}{P}$	$\frac{1a-1b}{P}$	$\frac{1}{t_1 t_2 t_3}$
$\frac{14a+14b}{P}$	$\frac{1a+1b}{P}$	$-\frac{1}{t_1 t_2 t_3}$
$\frac{14a-14b}{P}$	$\frac{1a+1b}{P}$	$\frac{s_1+s_3}{s_4(1+s_2)}$ (s for t_1, t_2, t_3)
$\frac{14a+14b}{P}$	$\frac{1a-1b}{P}$	$-\frac{s_1+s_3}{s_4(1+s_2)}$ (")







From the various values of the elements

form sets of parallels with [illegible]

[illegible]

[illegible]

groups are given

below. The group with element $\frac{1}{t_1 t_2 t_3}$ is

made up of the points of the following

pairs: $\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$, $\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$, $\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$, $\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$, $\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$, and $\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$;

The group with element $\frac{1}{t_1 t_2 t_4}$ is

$\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$, $\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$, $\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$, $\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$, $\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$, and $\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$;

[illegible] $\frac{1}{t_2 t_3 t_4}$

$\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$, $\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$, $\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$, $\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$, $\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$, and $\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$;

[illegible] $\frac{1}{t_2 t_3 t_4}$ is

$\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$, $\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$, $\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$, and $\left(\frac{u+v}{P}, \frac{u+w}{P}\right)$.

Any perpendicular to a line of a group is

Obtain an example by changing the sign between the letters of the numerators of the fractions, leaving the denominators as they are. Thus, $\frac{1}{1^{14}} - \frac{1}{2^{14}} + \frac{1}{3^{14}} - \frac{1}{4^{14}}$ is the sum of a series of fractions $\frac{1}{1^{14}} + \frac{1}{2^{14}} + \frac{1}{3^{14}} + \frac{1}{4^{14}}$. Similarly, corresponding to every series of fractions, one can find another series of fractions, such that $\frac{1}{1^{14}} + \frac{1}{2^{14}} + \frac{1}{3^{14}} + \frac{1}{4^{14}} = \frac{1}{1^{14}} - \frac{1}{2^{14}} + \frac{1}{3^{14}} - \frac{1}{4^{14}}$. Hence, we have 16 pairs of such corresponding series. Thus, in all, there are

two sorts of fractions of positive sign;

" " corresponding to them " negative sign;

six sorts of fractions in all.

These sorts of fractions may be considered as accounting for all of relations on the two sets of corresponding numbers arising from the decimal

As was noted above, there are associated with each of the four [illegible] circles four complete [illegible], obtained by joining up in all ways the four intersections, two and two, of a pair of circles of opposite sets, [illegible] [illegible] of [illegible] [illegible] the diagonals [illegible] [illegible] [illegible] themselves; [illegible] that four [illegible] points are the two intersections of each [illegible]tic circle by $34x - [illegible]$, [illegible], [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] each of the [illegible] [illegible]. In other words if the intersections of the [illegible] circle [illegible], the [illegible] [illegible] [illegible] the intersections of [illegible] [illegible] [illegible] all pass through the same point. And so for every other [illegible] of the [illegible] [illegible]

Thus ω permit for the circle corresponding to ω - infinite hexatactic ray, a unicursal parameter, θ, in the standard manner $\left(\frac{\omega\theta}{\omega}\right)$. This point is also evidently the mate of the four vertices of the original elliptic ω from circle of the hyperbolic ω.

Again, by joining the two intersections of an elliptic circle by the lines of centers to those on a hyperbolic circle, four vertices of the complete four-side are put on these circles. It happens, however, that the other two vertices are precisely the two intersections of these circles. In other words, the [illegible] of a complete four-side inscribed in any two conics at the [illegible] intersections of the two conics. That this limits on four-circles has already been shown. The rest of the statement is easily proved. Take the circles ω and $1R$. The locus of [illegible]

give the four intersections $\frac{u \pm v}{p}$ and $\frac{u' \pm v'}{p}$ with abscissæ $\frac{1}{t_2 t_3 t_4}$, $-\frac{S_1+S_3}{S_4(1+S_2)}$, $-\frac{1}{t_2 t_3 t_4}$, $\frac{S_1+S_3}{S_4(1+S_2)}$, in which the S's refer to t_2, t_3, t_4. The point of contact of chord $\frac{u+v}{p}\frac{u'+v'}{p}$ is given by

$$X - \frac{1-n}{S_4(m+n)}\overline{X} = \frac{2mn}{p(m+n)}.$$

The equidistant point of the tangents of the parabola with abscissæ $\frac{1}{t_2^2}, \frac{1}{t_3^2}, \frac{1}{t_4^2}$ were

$$X = \frac{1+S_2}{(1+S_1+S_2+S_3)(1-S_1+S_2-S_3)} = \frac{1+S_2}{BA}, \quad A+B \text{ for } 3t$$

$$\overline{X} = \frac{-S_3(S_1+S_3)}{BA} \quad " \quad "$$

Now, the point of $\frac{u+v}{p}$ and $\frac{u'+v'}{p}$ has the abscissa $t_2 t_3 t_4 = \frac{1}{S_4}$. Substituting in the equation of contact and clearing of fractions, we have

$$-\frac{1+S_2}{AB} + \frac{1}{S_4}\left(\frac{S_1+S_3}{AB}\right) =$$

$$\frac{2(1+S_1+S_2+S_3+S_4)(1+S_1+S_2+S_3)(1-S_1+S_2-S_3)}{(1+S_1+S_2+S_3+S_4)(1-S_1+S_2-S_3)(1+S_1-S_2+S_3)(1-S_1-S_2+S_3)}$$

or $\frac{1+S_2}{AB} + \frac{S_1+S_3}{AB} = \frac{B}{AB}$. Whence $\frac{1}{A} = \frac{1}{A}$, and the equation of the locus is satisfied. The equidistant points of the lines given by t_2^2, t_3^2, t_4^2 is in the form of $\frac{u+v}{p}$ and $\frac{u'+v'}{p}$. But this equidistant point is not

intersections of this circle & ... in IR. The
line each element $-\frac{1}{t_1 t_2 t_4}$ also ... on the
circumference as can be similarly shown.
Thus each pair of circles ... have ...
common tangent ... the ... intersection
of the two circles ...
In this manner, the other four circles, viz.,
the four-fold tangent circles

$$\chi + \frac{S_1+S_3}{S_4(1+S_2)}\bar{\chi} = \frac{2(a+b)(\acute{a}-\acute{b})}{P[(a+b)+(\acute{a}-\acute{b})]}$$

and $$\chi - \frac{S_1+S_3}{S_4(1+S_2)}\bar{\chi} = \frac{2(a-b)(\acute{a}+\acute{b})}{P[(a-b)+(\acute{a}+\acute{b})]}$$

intersect ... the point

$$\chi = \frac{\acute{t}}{(1-t_2^2)(1+S_2)}, \quad \bar{\chi} = \frac{-\frac{1}{2}\chi^2}{(1-t_2^2)(S_1+S_3)}.$$

The equations of the circles O and IR
are respectively

$$P^2\chi\bar{\chi} - aP(S_4\chi+\bar{\chi}) = S_4(b^2-a^2)$$

and $$P^2\chi\bar{\chi} - \acute{a}P(\acute{S}_4\chi+\bar{\chi}) = \acute{S}_4(\acute{b}^2-\acute{a}^2).$$

Combining these equations, the equation of condition of intersection is

$$-aP(S_4\chi+\bar{\chi}) + \acute{a}P(\acute{S}_4\chi+\bar{\chi}) = S_4(b^2-a^2) - \acute{S}_4(\acute{b}^2-\acute{a}^2),$$

where the a's, b's, and δ's refer to four t's. For t_2, t_3, t_4 the equation becomes on substituting the values of k and $\lambda = b\sqrt{-1}$, and reducing $2t_1\delta_1\left[(1-t_1^2)(1+\delta_1+\delta_2+\delta_3)(1-\delta_1+\delta_2-\delta_3)\right] = 2t_1\delta_1\left[(1-t_1^2)(1+\delta_1+\delta_2-\delta_3)(1-\delta_1+\delta_2+\delta_3)\right]$. Thus, the second pair of circles intersect at right angles at the intersections of the circles. Thus is the common solution of these circles; and it is seen that if each of the orthogonal intersections of one pair of circles is a focus of the four tangent circles of the other pair, one focus is the circumcenter and the other is $-\frac{1}{t_2t_3t_4}$, while the other two are $\frac{\delta_1+\delta_2}{\delta_4(1+\delta_3)}$ and $-\frac{\delta_1+\delta_2}{\delta_4(1+\delta_3)}$, δ_1, δ_2, and δ_3 referring to three t's.

Section 3.

The Orthocentric Circles of Four Tangent Lines of a Parabola.

We had the line equation of the parabola in the form

$$tx - \frac{y}{t} = \frac{t}{1-t^2}$$

with slope $\frac{1}{t^2}$. The slope of a perpendicular to this line is $-\frac{1}{t^2}$. The equidistant point of the three lines of the parabola whose slopes are $\frac{1}{t_1^2}$, $\frac{1}{t_2^2}$, $\frac{1}{t_3^2}$ is

$$x = \frac{1+S_2}{P} \qquad (S \text{ and } P \text{ for } 3\ t\text{'s})$$

or
$$2x = \frac{1}{1-S_1+S_2-S_3} + \frac{1}{1+S_1+S_2+S_3}$$

Symmetrizing this equation for four lines, we have

$$2x = \frac{1-t}{(1-t_1)(1-t_2)(1-t_3)(1-t_4)} + \frac{1+t}{(1+t_1)(1+t_2)(1+t_3)(1+t_4)}$$

or
$$2x = \frac{1-t}{A} + \frac{1+t}{B}$$

$$2\overline{x} = -\frac{(1-t)\frac{S_4}{t}}{A} + \frac{(1+t)\frac{S_4}{t}}{B}.$$

Now, a perpendicular from the equidistant

point of the three lines, under consideration upon a fourth line is of form

$$2x + \frac{2}{t_4^2}\bar{x} = k \quad ;$$

whence, dropping the subscript 4 and putting the perpendicular on the point

$$2x = \frac{1}{A} + \frac{1}{B}$$

the result is

$$2x + \frac{2}{t^2}\bar{x} = \frac{1}{A} + \frac{1}{B} - \frac{S_4}{t^2 A} + \frac{S_4}{t^2 B} .$$

Symmetrizing

$$2x + \frac{2}{t^2}\bar{x} = \frac{1-t}{A} - \frac{(1-t)\frac{S_4}{t^3}}{A} + \frac{1+t}{B} + \frac{(1+t)\frac{S_4}{t^3}}{B} ,$$

an equation embracing the four perpendiculars, one each from the four equidistant points of 3 of 4 lines. Simplifying the equation, we have

$$2t^2 x + 2\bar{x} = \frac{t^2(1-S_1) + S_2 t - S_3 + S_4}{A} + \frac{t^2(1+S_1) - S_2 t + S_3 + S_4}{B} ,$$

the line equation of the orthocentric curve for four curves of n dimensions. The map equation is

$$2x = \frac{1-S_1}{A} + \frac{1+S_1}{B} + \frac{S_2}{t}\left(\frac{1}{A} - \frac{1}{B}\right)$$

again, consider three lines given by t_1^2, t_2^2, $t_3^2 \delta_4$. Reverse line 1, calling these briefly lines $\hat{1}$, 2, 3. The center of the circle on the new set of four equidistant points is

$$2x = \frac{1-t}{A} + \frac{1+t}{B},$$

and therefore the line equation of the orthocentric circle for these lines is

$$2t^2 x + 2\bar{x} = \frac{t^2(1-\delta_1') + \delta_2 t - \delta_3 + \delta_4}{A} + \frac{t^2(1+\delta_1') - \delta_2 t + \delta_3 + \delta_4}{B}$$

and the map equation is

$$2x = \frac{1-\delta_1'}{A} + \frac{1+\delta_1'}{B} + \frac{\delta_2}{t}\left(\frac{1}{A} - \frac{1}{B}\right).$$

Similarly, the map equation for lines 1 and 2 reversed is

$$2x = \frac{1-\delta_1''}{A''} + \frac{1+\delta_1''}{B''} + \frac{\delta_2''}{t}\left(\frac{1}{A''} - \frac{1}{B''}\right).$$

Reversing the lines in all ways there are 8 different orthocentric circles, and it is a natural question to ask how their centers are related, since there evidently are relations inasmuch as four lines give rise to 8 circles.

Now, the cosine of the angle per unit length of the bisecting curve

$$2\chi = \frac{1-S_1}{A} + \frac{1+S_1}{B}.$$

As $A = 1 - S_1 + S_2 - S_3 + S_4$

and $B = 1 + S_1 + S_2 + S_3 + S_4$

$$A+B = 2(1+S_2+S_4)$$

and $B - A = 2(S_1 + S_3)$

Therefore, $2AB\chi = 1+S_2+S_4 - S_1(S_1+S_3)$,

$$2A'B'\chi = 1+S_2'+S_4' - S_1'(S_1'+S_3')$$

$$2\bar{A}\bar{B}\chi = 1+\bar{S}_2+\bar{S}_4 - \bar{S}_1(\bar{S}_1+\bar{S}_3).$$

Similar expressions for cosines per unit length of the corresponding curves $x'x'$, $x''x''$, etc. Hence

$$AB\chi = A'B'\chi = \bar{A}\bar{B}\bar{\chi},$$

etc., i.e., the products $AB\chi$, $A'B'\chi'$, etc. are constants so far as the change of signs of the t's is concerned.

If now we eliminate from the equation of the cosines of the [illegible]

find the symmetric functions of the t_1^2, we may be able to get a relation on the coefficients by interpreting properly the symmetric functions of this latter which is a rational function of the t_1, t_2, t_3, t_4. If this is done, further, suppose the coefficients of powers of x in the equation that we obtain immediately follows we shall use as such S_2 for $+ t_1't_2'$. From the relation

$$t^4 - S_1 t^3 + S_2 t^2 - S_3 t + S_4 = 0$$

we get $(t^4 + S_2 t^2 + S_4)^2 = (S_1 t^3 + S_3 t)^2$

or $t^8 + (S_2^2 - 2S_4 - S_1^2) t^6 + (S_4^2 - S_1^2 S_2 + 2S_2 S_4 - S_3^2) t^4 + (S_2 S_4^2 - S_1^2 S_4 - S_3^2 S_2) t^2 + S_4^2 = 0$

or $t^8 - \sigma_1 t^6 + \sigma_2 t^4 - \sigma_3 t^2 + \sigma_4 = 0$.

Here the σ's are symmetric functions of $t_1^2, t_2^2, t_3^2, t_4^2$, and are consequently unaffected by simply changing the signs of t_1, t_2, t_3, t_4; and by changing the signs of these t's we get the symmetric functions of the t's with the opposite signs corresponding, [illegible]

the angles obtained in terms of the sides of the triangle. From the preceding equations we get at once the following relations between the σ's and the S's:

(a) $\sigma_1 = S_1^2$

(b) $\sigma_2 = S_2^2 + 2S_4 - 2S_1S_3$

(c) $\sigma_3 = S_3^2 - 2S_2S_4$

(d) $\sigma_4 = S_4^2$

In the equation, $ABX = 1 + S_2 + S_4 - S_1^2 - S_1S_3$, substitute $-2S_2 - \sigma_1$ for $-S_1^2$ and for $-S_1S_3$ $-\frac{1}{2}\sigma_2 - \frac{1}{2}S_2^2 - S_4$. We then obtain

$$ABX = 1 - \sigma_1 + \frac{1}{2}\sigma_2 - S_2 - \frac{1}{2}S_2^2,$$

an expression for the number of the congruent orthocentric quads, but in which ABX, the σ_1, and S_4 are not constant for a change of angle of either one of the σ_2 or S_2 at a time, the fact that the σ's are always constants, but S_4 is subject to the limitation just mentioned.

S_2 and S_2^2, of course, vary with a change of sign of the t's.

Hence, if S_2 is considered a parameter, the equation $\#BX = 1 - \sigma_1 + \sigma_2 - S_2 - \frac{1}{2}S_2^2$ represents a curve on the cubic of the original circle, but similarly for the other expressions, for instance, $\#BX = 1 - \sigma_1 + \frac{1}{2}\sigma_2 - \frac{13}{S_2} - \frac{1}{2}\left(\frac{13}{S_2}\right)^2$ is a curve on the cubic of the orthocentric circle for lines 1 and 3 reversed. It is natural to divide the eight orthocentric circles under consideration into two groups, — that for a change in direction of only one line at a time and that for a change of two simultaneously, together with the original circle. This is for because S_4 is constant, but a different constant for each change. Now, the conjugate of the parameter S_2 is $\frac{2}{S_4}$, that is, $\overline{S_2} = \frac{S_2}{S_4}$

or for a [illegible] S_2, [illegible] it X,

$$S_4 k - \bar{\gamma} = 0 .$$

But we recognize this as the first problem of the elliptic set of Jacobi evaluated for the functions. So does not change sign for a reversal of both terms, but does change sign for a reversal of real part. That is, for a change of direction of both terms, so same as the line

$$\bar{S}_4 k + \bar{\gamma} = 0$$

in the k-plane. But terms of modulus of the hyperbolic set of Jacobi evaluated for the terms, hence, ν_- is not the same as the terms for the line $S_4 k \pm \bar{\gamma} = 0$.

Conclusion: (a) $\#B_k = 1 - \sigma_1 + \frac{1}{2}\sigma_2 - \bar{S}_2 - \frac{1}{2}\left(\frac{1}{2}\right)^2$

(b) $\#B_k = 1 - \sigma_1 + \frac{1}{2}\sigma_2 - \bar{S}_2 - \frac{1}{2}\left(\frac{1}{2}\right)^2$

(c) $\#B_k = 1 - \sigma_1 + \frac{1}{2}\sigma_2 - \frac{3}{2} - \frac{1}{2}\left(\frac{3}{2}\right)^2$

(d) $\#B_k = 1 - \sigma_1 + \frac{1}{2}\sigma_2 - \frac{11}{2} - \frac{1}{2}\left(\frac{4}{2}\right)^2$.

(a) S_2 is the one of the terms $S_4 k + \bar{\gamma} = 0$, the terms that occur in the line from the

to the [illegible] to [illegible] the second degree [illegible] evidently [illegible] a conic. This conic is on the 4 curves when S_2 assumes the values $\overset{1}{S}_2$, $\overset{2}{S}_2$, $\overset{3}{S}_2$, $\overset{4}{S}_2$. To determine the kind of conic, it is only [illegible] to [illegible]

$$w = z + \tfrac{1}{2} z^2$$

where w moves [illegible],
and z moves in a [illegible] in the [illegible].
Let $w = u + iv$ and $z = x + iy$,
then $u + iv = x + iy + \frac{1}{2}(x^2 - y^2 + 2ixy)$ and
consequently $u = \frac{1}{2}(x^2 - y^2) + x$

[illegible] $v = y(1 + x)$

But z [illegible] the line $y = mx$, then

$$v = mx + [illegible]$$

$$x = -\tfrac{1}{2} \pm \sqrt{\frac{m + 4v}{4m}},$$

then $2u = -1 \pm \sqrt{\frac{m+4v}{4m}} + \frac{1}{4} \mp 2\sqrt{\frac{m+4v}{4m}} + \frac{m+4v}{4m}$

or, rationalizing

$$4u^2 - 4(\tfrac{1}{m} - m)uv + (\tfrac{1}{m} - m)v^2 \ldots [illegible] \ldots - \tfrac{1}{4} [illegible]^2$$

If $A = 4$, $B = (\frac{1}{m} - m)^2$, $2H = -4(\frac{1}{m} - m)$, etc.

being $2B - H^2 = 3$ [illegible]

[illegible] that [illegible] is [illegible] parabola. [illegible] from the [illegible] on the [illegible], a particular parabola on the [illegible] [illegible].

In a similar [illegible], we can show that the [illegible] [illegible] with four values

$$(e)\ ABx = 1 - \sigma_1 + \tfrac{1}{2}\sigma_2 - s_2 - [illegible]$$

$$(f)\ ABx = 1 - \sigma_1 + \tfrac{1}{2}\sigma_2 - s_2^{12} - [illegible]$$

$$(g)\ ABx = 1 - \sigma_1 + \tfrac{1}{2}\sigma_2 - s_2^{13} - [illegible]$$

$$(h)\ ABx = 1 - \sigma_1 + \tfrac{1}{2}\sigma_2 - s_2^{14} - [illegible]$$

of the parabola

$$ABx = 1 - \sigma_1 + \tfrac{1}{2}\sigma_2 - [illegible]$$

where s_2, the parameter, [illegible] four values indicated for it in (e), (f), (g), (h) and runs on the line

$$s_4 x - \bar{x} = 0,$$

the locus of centers of the complete set of the circles for 4 lines. We may [illegible]

$$AB\chi = 1 - \sigma_1 + \tfrac{1}{2}\sigma_2 - s_2 - \tfrac{1}{2}s_2^2$$

is a parabola by eliminating the parameter s_2. Consider this curve for s_4 positive.

Then (a) $$s_2^2 + 2s_2 = 2 - 2\sigma_1 + \sigma_2 - 2AB\chi$$

$$s_2 = -1 \pm \sqrt{3 - 2\sigma_1 + \sigma_2 - 2AB\chi}$$

The conjugate of (a) is

$$s_2^2 + 2s_4 s_2 = 2\sigma_4 - 2\sigma_3 + \sigma_2 - 2AB\bar{\chi}$$

Substituting in this equation the value of s_2 just found, using either the positive or negative sign before the radical, we have

$$A^2B^2(\chi-\bar{\chi})^2 + \left[2AB(s_4-1) - 2AB(1-\sigma_1+\sigma_3-\sigma_4)\right](\chi-\bar{\chi})$$

$$+ 2AB(s_4-1)^2\chi - (2-2\sigma_1+\sigma_2)(s_4-1)^2 - 2(s_4-1)(1-\sigma_1+\sigma_3-\sigma_4) + (1-\sigma_1+\sigma_3-\sigma_4)^2 = 0,$$

which evidently is the equation of a parabola. This arranged with reference to $\bar{\chi}$ is:

$$\bar{\chi}^2 - \left[2\chi + \frac{\left[2(s_4-1) - 2(1-\sigma_1+\sigma_3-\sigma_4)\right]}{AB}\right]\bar{\chi} + \chi^2 + \left[\frac{2s_4(s_4-1) - 2(1-\sigma_1+\sigma_3-\sigma_4)}{AB}\right]\chi$$

$$- \frac{2(s_4-1)\left[s_4(1-\sigma_1+\tfrac{1}{2}\sigma_2) - (\sigma_4 - \sigma_3 + \tfrac{1}{2}\sigma_2)\right]}{A^2B^2} + \frac{(1-\sigma_1+\sigma_3-\sigma_4)^2}{A^2B^2} = 0.$$

The discriminant as to $\bar{\chi}$ gives the foci. Hence,

$$4\chi^2 + \frac{\left[8(s_4-1) - 8(1-\sigma_1+\sigma_3-\sigma_4)\right]}{AB}\chi + \frac{\left[2(s_4-1) - 2(1-\sigma_1+\sigma_3-\sigma_4)\right]^2}{A^2B^2} =$$

$$4\chi^2 + \frac{\left[8(s_4-1) - 8(1-\sigma_1+\sigma_3-\sigma_4)\right]}{AB}\chi - \frac{8(s_4-1)\left[(s_4-1) + \tfrac{1}{2}\sigma_2 - (\sigma_4 - \sigma_3 + \tfrac{1}{2}\sigma_2)\right]}{A^2B^2} + \frac{4(1-\sigma_1+\sigma_3-\sigma_4)^2}{A^2B^2}$$

The root of this equation is infinite, the other being

$$\frac{2(S_4-1)^2}{AB}\chi = \frac{(S_4-1)^2}{A^2B^2} - \frac{2(S_4-1)(1-\sigma_1+\sigma_3-\sigma_4)}{A^2B^2} + \frac{2(S_4-1)\left[S_4(1-\sigma_1+\frac{1}{2}\sigma_2)+\sigma_4-\sigma_3+\frac{1}{2}\sigma_2\right]}{A^2B^2}$$

$$\text{or} \qquad \chi = \frac{1+2(1-\sigma_1+\frac{1}{2}\sigma_2)}{2AB}$$

If we form the $\bar{\chi}$-discriminant of the parabola for S_4 negative, the focus is given by

$$\frac{2(S_4+1)^2}{AB}\chi = \frac{(S_4+1)^2}{A^2B^2} + \frac{2(S_4+1)^2(1-\sigma_1+\frac{1}{2}\sigma_2)}{A^2B^2}$$

$$\text{or} \qquad \chi = \frac{1+2(1-\sigma_1+\frac{1}{2}\sigma_2)}{2AB},$$

but this was the focus of the parabola for S_4 positive. The two parabolas on which are the four centers of the two sets of orthocentric circles for your desirable curve of the original parabola are, then, confocal. Moreover as the parabolas are confocal, they intersect orthogonally.

It will be shown later that these two center parabolas intersect at the point

$$\chi = \frac{1-\sigma_1+\frac{1}{2}\sigma_2}{AB} \qquad \bar{\chi} = \frac{\sigma_2-\sigma_3+\frac{1}{2}\sigma_4}{AB}.$$

Where is the other intersection? The Cartesian

equation of the conic is obtained from the fraction

is $y^2 + \left[c - \bar{c} - \frac{S_4^2 - 1}{2AB} i\right] y - \frac{(S_4 - 1)\bar{c}}{2AB} X + \frac{S_4 - 1}{AB} \bar{c} - c\bar{c} = 0$

where $c \equiv \frac{1 - \sigma_1 + \frac{1}{2}\sigma_2}{AB}$ and $\bar{c} \equiv \frac{\sigma_4 - \sigma_3 + \frac{1}{2}\sigma_2}{AB}$,

the parabola for P_4 as vertex is on the X, Y plane

$$y^2 + \left[c - \bar{c} - \frac{S_4^2 - 1}{2AB} i\right] y - \frac{(S_4 + 1)^2}{2AB} X + \frac{(S_4 + 1)\left[S_4 c + \bar{c}\right]}{2AB} - \frac{c - \bar{c}}{2AB} = 0.$$

Combining these equations, we have

$$X = \tfrac{1}{2}(c + \bar{c}), \quad Y = -\tfrac{1}{2}(c - \bar{c})i$$

which transformed to conjugate variables

is $\quad x = \bar{c}; \quad \bar{x} = \bar{c}.$

Briefly written, the above two equations are

$$y^2 + ay - bX + d - \ell = 0$$

and $\quad y^2 + a'y - b'X + d' - \ell = 0,$

whence, as the parabolas are on the point $\frac{1}{2}(c + \bar{c})$

and the sum of the roots is $-a \equiv -(c - \bar{c})i + \frac{(S_4^2 - 1)i}{2AB}$,

the other point of intersect is

$$Y = -\frac{1}{2}(c - \bar{c})i + \frac{(S_4^2 - 1)i}{2AB}.$$

Therefore the two parabolas intersect in the

point (conjugate coordinates)

$x = c, \ \bar{x} = \bar{c}$; $X = c - \frac{S_4^2 - 1}{2AB}$, $\bar{X} = \bar{c} + \frac{S_4^2 - 1}{2AB}$, and

the line on the formula, $x + \bar{x} = c + \bar{c}$, is parallel to the axis of imaginaries.

If, according to the abbreviations adopted, $\frac{1 - \sigma_1 + \frac{1}{2}\sigma_2}{AB} \equiv c$, $\frac{\sigma_4 - \sigma_3 + \frac{1}{2}\sigma_2}{AB} \equiv \bar{c}$, $\frac{1 - \sigma_1 + \sigma_3 - \sigma_4}{AB} \equiv c - \bar{c}$, we use the polar operator $x\frac{\partial}{\partial x_1} + \bar{x}\frac{\partial}{\partial \bar{x}_1} + z\frac{\partial}{\partial z_1}$ on the equation of the parabola for s_4 positive,

$$\bar{x}^2 - 2x\bar{x} - 2\left(\frac{s_4 - 1}{AB}\right)\bar{x}z + 2(c - \bar{c})\bar{x}z + x^2 + 2s_4\left(\frac{s_4 - 1}{AB}\right)xz - 2(c - \bar{c})xz$$
$$- \left[2\left(\frac{s_4 - 1}{AB}\right)s_4 c - 2\left(\frac{s_4 - 1}{AB}\right)\bar{c} - (c - \bar{c})\right]z^2 = 0,$$

we get $-x\left[2x_1 - 2\bar{x}_1 + 2s_4\left(\frac{s_4 - 1}{AB}\right) - 2(c - \bar{c})\right] + \bar{x}\left[2\bar{x}_1 - 2x_1 - \left(\frac{s_4 - 1}{AB}\right) + 2(c - \bar{c})\right]$
$+ z\left[-2\left(\frac{s_4 - 1}{AB}\right)\bar{x}_1 + 2(c - \bar{c})\bar{x}_1 + 2s_4\left(\frac{s_4 - 1}{AB}\right)x_1 - 2(c - \bar{c})x_1 - 2z_1\left(\frac{s_4 - 1}{AB}\right)s_4 c - 2\left(\frac{s_4 - 1}{AB}\right)\bar{c} - (c - \bar{c})\right]$
$= 0$. The tangent at $x = c$, then, is

$$s_4 x - \bar{x} = s_4 c - \bar{c}$$

Similarly, the equation of the tangent at $x = c$ to the parabola for s_4 negative is

$$s_4 x + \bar{x} = s_4 c + \bar{c}.$$

These tangents are perpendicular, thus verifying the result already obtained that the parabola intersects at right angles, at the point c. The polar of the focus of the parabola for

S_4 positive, the focus being $\frac{1}{2AB} + c$, is

$$k + \bar{k} = c + \bar{c} + \frac{S_4}{AB};$$

for the parabola for S_4 negative it is

$$k + \bar{k} = c + \bar{c} - \frac{S_4}{AB}.$$

That is, the two directrices of the center-parabolas are parallel to the axis of imaginaries. The common axis is

$$k - \bar{k} = \frac{1}{2AB} + c - \frac{S_4^2}{2AB} - \bar{c}.$$

Thus, the centers of the eight orthocentric circles for four directed lines of a parabola are four each on two center-parabolas. These two [illegible] $k + \frac{S_4}{2AB}$, [illegible] are not of the same size, and are* similar and similarly placed with reference to [illegible].

Consider next the equation of the orthocentric circle itself, viz.,

$$ABx - 1 + \sigma_1 - \tfrac{1}{2}\sigma_2 + S_2 + \tfrac{1}{2}S_2^2 = \frac{S_2}{2t}\left(S_1 + S_3\right),$$

*Salmon. Conic Sections, p. 223.

where for convenience s_1 and s_4 have not been expressed in terms of the σ_i. The conjugate of this equation is

$$AB\overline{X} - \sigma_4 + \sigma_3 - \tfrac{1}{2}\sigma_2 + s_2 s_4 + \tfrac{1}{2}s_2^2 = \tfrac{1}{2}s_2 t(s_1 + s_4).$$

The product of these two equations, member by member, is

$$(ABX - 1 + \sigma_1 - \tfrac{1}{2}\sigma_2 + s_2 + \tfrac{1}{2}s_2^2)(AB\overline{X} - \sigma_4 + \sigma_3 - \tfrac{1}{2}\sigma_2 + s_2 s_4 + \tfrac{1}{2}s_2^2) = \tfrac{1}{4}s_2^2(s_1+s_4)^2$$

But $$s_1^2 = \sigma_1 + 2s_2,$$

$$2\sigma_3 = -\sigma_2 + s_2^2 + 2s_4,$$

and $$s_4^2 = \sigma_3 + 2s_2 s_4.$$

Substituting these values in right-hand member, we have —

$$\tfrac{1}{4}s_2^2(s_1+s_4)^2 = \tfrac{1}{4}s_2^2\sigma_1 + \tfrac{1}{2}s_2^2 - \tfrac{1}{4}s_2^2\sigma_2 + \tfrac{1}{4}s_2^4 + \tfrac{1}{2}s_2^2 s_4 + \tfrac{1}{4}[illegible]$$

$+ \frac{1}{2}s_2^2 s_4$. Also, the product of the equations in left member is

$$A^2B^2 X\overline{X} - (1 - \sigma_1 + \tfrac{1}{2}\sigma_2 - s_2 - \tfrac{1}{2}s_2^2)(AB\overline{X} - \sigma_4 + \sigma_3 - \tfrac{1}{2}\sigma_2 + s_2 s_4 + \tfrac{1}{2}s_2^2)ABX$$

$+ \sigma_4 - \sigma_1\sigma_4 + \frac{1}{2}\sigma_2\sigma_4$ [illegible]

$+ \frac{1}{4}$ [illegible]

$-$ [illegible]

We have here eliminated t_3 as a parameter, and have expressed the product in terms of the σ's, S_4 (for a set of 4 circles), and S_2, which may used as a parameter. The σ's and σ_4 for a set of 4 circles are unchanged with the four as a change of sign of one or two of t_1, t_2, t_3 and t_4 is concerned. Moreover S_2 does not enter to a degree higher than the second. The numerical giving this location of the expression, [illegible] the form

$$\left[2A^2B^2\chi\bar{\chi} - 2AB\bar{\chi} - \sigma_4 AB\chi + 2\sigma_1 AB\bar{\chi} + \sigma_1\sigma_4 AB\chi - \sigma_2 AB\bar{\chi} - \sigma_2 AB\chi + 2\sigma_4 - 2\sigma_1\sigma_4 + \sigma_2\sigma_4 - 2\sigma_3 + 2\sigma_1\sigma_3 - \sigma_2\sigma_3 + \sigma_2 - \sigma_1\sigma_2 + \tfrac{1}{2}\sigma_2^2\right] + 2S_2\left[AB\bar{\chi} + S_4 AB\chi - \sigma_4 + \sigma_3 - \tfrac{1}{2}\sigma_2 - S_4 + \sigma_1 S_4 - \tfrac{1}{2}\sigma_2 S_4\right] + S_2^2\left[AB\bar{\chi} + AB\chi - \sigma_4 + \tfrac{1}{2}\sigma_3 - \tfrac{1}{2}\sigma_2 + \sigma_4 - 1 + \tfrac{1}{2}\sigma_1\right] = 0$$

This whole expression is self-conjugate as well as each part included in the parentheses. The first part whose first term is $2A^2B^2\chi\bar{\chi}$ is evidently a conic since the conjugate

coordinates x and $\bar{x}$ enter into a combination of only the second degree. Also, the terms x^2 and $\bar{x}^2$ are missing thus indicating the presence of two infinite roots and showing that the conic, speaking projectively, is on the circular points. Whence the conic is a circle. It is in fact a point circle. The second and third parts are evidently lines, which, however, in our inversive geometry are circles. Thus the equation is of the form

$$\lambda^2 U + 2\lambda V + W = 0$$

in which U, V, W are circles. Hence the whole equation represents a quadratic system of circles, and this in contrast to the linear system of Lond circles for the four directed lines. A circle, for instance, of the hyperbolic set was $x\bar{x} + \frac{s_4(1-s_4)}{AB}x - \frac{1-s_4}{AB}\bar{x} - \frac{s_4}{AB} + s_2\left[\frac{(s_4 x - \bar{x})}{AB}\right] = 0$,

where the parameter t has been eliminated. If s_2 is considered as a parameter, this circle is one of the pencil

$$U + \lambda V = 0,$$

in which $U \equiv x\bar{x} + \frac{s_4(1-s_4)}{AB} - \frac{1-s_4}{A}\bar{x} - \frac{s_4}{B} = 0$ and $V \equiv s_4 x - \bar{x} = 0$.

The four orthocentric circles of one set belong to one quadratic system, while the four circles of the other set belong to another system.

In the circle $x = a + bt$, or in conjugate coordinates

$$x\bar{x} - a\bar{x} - \bar{a}x + a\bar{a} + b\bar{b} = 0,$$

the center is given by the negative of the coefficient of $\bar{x}$ when the coefficient of $x\bar{x}$ is unity. In the quadratic system of circles just given when — given any s_2 one has a circle; but if given any x, one has two values of s_2, that is, the

are two circles on a point. Further, given s_2, the center of the circle is the negative of the coefficient of $\bar{x}$, which, in this instance, is

$$\frac{1 - \sigma_1 + \frac{1}{2}\sigma_2 - s_2 - \frac{1}{2}s_2^2}{\Delta 13}.$$

But $\Delta BX = 1 - \sigma_1 + \frac{1}{2}\sigma_2 - s_2 - \frac{1}{2}s_2^2$ was the parabola which contained the centers of the four circles of one set of the two into which the eight orthocentric were divided. Thus, the center of every circle of the quadratic system under consideration is on one of the two parabolas

(a) $\Delta BX = 1 - \sigma_1 + \frac{1}{2}\sigma_2 - s_2 - \frac{1}{2}s_2^2$

or (b) $\Delta BX = 1 - \sigma_1 + \frac{1}{2}\sigma_2 - s_2 - \frac{1}{2}(s_2)^2$.

The four circles including the original circle and those for a change of direction of two lines at a time are on (a); those for a change of one line are on (b).

Now, bicircular quartics may be regarded

as the envelope of the quadratic system of circles given by*

$$\lambda^2 U + 2\lambda V + W = 0$$

Again, all circles included in the form

$$\alpha U + \beta V + \gamma W = 0$$

have a common orthogonal circle, while the centers of the circles of the system

$$\lambda^2 U + 2\lambda V + W = 0$$

are on a conic. Hence, the bicircular quartic may be regarded as the envelope of a circle whose center moves on a fixed conic F, say, and which cuts a fixed circle orthogonally.

However, if the conic F touch the line joining the two common points of the circles, the quartic breaks down into a circular cubic and the line at infinity.

In the present case, we have shown

*See Salmon, H.P. Curves (Reach), page 339 et seq.

that the orthocentric circle belongs to a quadratic system, that is the circles of this system have their centres on a fixed line, which touches the line joining the two common points of the circles, corresponding to σ_1, the envelope of the quadratic system of circles will be a parabola with the line at infinity for its directrix. The other two of the four intersections of the circles will lie on the line at infinity, being respectively the two focal points of the parabola. Hence the circles of the system touch the line at infinity.

We must see also that these two systems of circles have the same envelope.

The equation, in conjugate coordinates, of the orthocentric circle may be written

$$2\left[x\bar{x} - \frac{(1-\sigma_1+\frac{1}{2}\sigma_2)}{AB}\bar{x} - \frac{(\sigma_4-\sigma_1+\frac{1}{2}\sigma_3)}{AB}x + \frac{(1-\sigma_1+\frac{1}{2}\sigma_2)(\sigma_4-\sigma_1+\frac{1}{2}\sigma_3)}{A^2B^2}\right]$$

$$+\frac{2S_2}{AB}\left[S_4\chi+\overline{\chi}-\frac{S_4(1-\sigma_1+\frac{1}{2}\sigma_2)}{AB}-\frac{(\sigma_4-\sigma_3+\frac{1}{2}\sigma_2)}{AB}\right]$$

$$+\frac{S_2^2}{AB}\left[\chi+\overline{\chi}-\frac{\sigma_4-\sigma_3+\frac{1}{2}\sigma_2}{AB}-\frac{1+\frac{1}{2}\sigma_3-\frac{1}{2}\sigma_1-S_4}{AB}\right]=0$$

Let $\frac{1-\sigma_1+\frac{1}{2}\sigma_2}{AB}\equiv c$, $\frac{\sigma_4-\sigma_3+\frac{1}{2}\sigma_2}{AB}\equiv\overline{c}$,

and $\frac{1+\frac{1}{2}\sigma_3-\frac{1}{2}\sigma_1-S_4}{AB}\equiv k$,

Then the circle may be written briefly

$$2\left[\chi\overline{\chi}-c\overline{\chi}-\overline{c}\chi+c\overline{c}\right]+\frac{2S_2}{AB}\left[S_4\chi+\overline{\chi}-S_4c-\overline{c}\right]+\frac{S_2^2}{AB}\left[\chi+\overline{\chi}-\overline{c}-k\right]=0,$$

where S_2 is the parameter and $\overline{c}+k$ is real.

Let $\chi\overline{\chi}-c\overline{\chi}-\overline{c}\chi+c\overline{c}\equiv W$,

$S_4\chi+\overline{\chi}-S_4c-\overline{c}=V$,

and $\chi+\overline{\chi}-\overline{c}-k=U$.

These equations are respectively a point circle, a line inclined to the axes of reals, and a line perpendicular to the axis. These are the 1-axe circles of the pencil $\lambda^2U+2\lambda V+W=0$. The envelope of this system is $UW=V^2$, which in the present case is

$$2AB\left[\chi\overline{\chi}-\overline{c}\chi-c\overline{\chi}+c\overline{c}\right]\left[\chi+\overline{\chi}-(\overline{c}+k)\right]=\left[S_4\chi+\overline{\chi}-S_4c-\overline{c}\right]^2.$$

This is plainly a self-conjugate equation, as

it should be. Expanded it is:

$$X^2\bar{X} + X\bar{X}^2 - \left[c + \bar{c} + (\bar{c}+k) + \frac{S_4}{AB}\right]X\bar{X} - \left(\bar{c} + \frac{S_4^2}{2AB}\right)X^2$$
$$-\left(c + \frac{1}{2AB}\right)\bar{X}^2 + \left[c\bar{c} + \bar{c}(\bar{c}+k) + \frac{S_4^2 c + S_4\bar{c}}{AB}\right]X$$
$$+\left[c\bar{c} + c(\bar{c}+k) + \frac{\bar{c} + S_4 c}{AB}\right]\bar{X} - \left[c\bar{c}(\bar{c}+k) + \frac{(cS_4 + \bar{c})^2}{2AB}\right] = 0.$$

The coefficient of $X\bar{X}$ is real, so is the constant term, while the coefficients of X^2 and $\bar{X}^2$ are conjugate.

In Cartesians the equation is:

$$X^3 + Xy^2 - \left[\bar{c} + \frac{S_4^2}{2AB} - c - \frac{1}{2AB}\right]iXy - \frac{1}{2}\left[2c + 2\bar{c} + (\bar{c}+k) + \frac{(S_4+1)^2}{2AB}\right]X^2$$
$$-\frac{1}{2}\left[(\bar{c}+k) - \frac{(S_4-1)^2}{2AB}\right]y^2 + \frac{1}{2}\left[2c\bar{c} + (c+\bar{c})(\bar{c}+k) + \frac{S_4^2\bar{c} + S_4(c+\bar{c}) + \bar{c}}{AB}\right]X$$
$$+\frac{1}{2}\left[(\bar{c} - c)(\bar{c}+k) + \frac{S_4^2 c + S_4(c-\bar{c}) - \bar{c}}{AB}\right]iy - \frac{1}{2}\left[c\bar{c}(\bar{c}+k) + \frac{(cS_4 + \bar{c})^2}{2AB}\right] = 0$$

The equation, in conjugate coördinates, just written is of form $UW = V^2$ in which U, V, and I are the base curves of the quadratic system; but as the envelope here is a degenerate case of the bicircular quartic, that is, a circular cubic and the line at infinity, we may write homogeneously

$$UW\bar{I} = I^2V^2, \text{ or } UW = IV^2,$$

where I is the line at infinity. The focus

this equation shows that the line U touches the cubic where U intersects the line V and intersects the cubic where U meets the line at infinity. This accounts for all the intersections of U with the cubic. Moreover the intersection of U with I accounts for two intersections on the quartic envelope, that is, U touches the quartic twice, as it should. It is evident that U is parallel to an asymptote, and we have seen it is parallel to the axis of symmetry.

$UW = IV^2$ also shows that the conic W (a point circle) touches the cubic at the intersections of V with W. As W is a point circle and as we must account for 6 intersections with the cubic or 8 with the whole envelope, we are led to suspect that the meet of W and V is a singular point on the envelope. This meet is the point $\frac{1}{A} - \frac{1}{B} + \frac{1}{C} = 0$, since at that point $V = \frac{1}{A}x + \frac{1}{B}y - \frac{1}{C}z - 1 = 0$ and also $W =$

$X\bar{Y} - c\bar{X} - \bar{c}X + z\bar{c} = 0$. Moreover c is the center of this circle.

It was noted above that U is parallel to an asymptote of the cubic. This is also seen from the Cartesian equation of the cubic, written for shortness, as

$$x^3 + xy^2 - \alpha x y - \beta x^2 - \gamma y^2 + \delta x + \epsilon y + \zeta = 0.$$

The highest power of y is here absent, thus indicating the presence of an infinite root for every finite value of x. The coefficient of y^2 is $x - \gamma$, that is, $x = \gamma$ is the asymptote parallel to the y-axis, verifying the statement above as to this asymptote. Next,

$$y = \frac{1}{2}\left[(\alpha + k) - \frac{(\alpha - k)^2}{2\beta}\right] = \frac{1}{4},$$

and is $x = \frac{1}{2}\left[(\alpha + k) - \frac{(\alpha - k)^2}{2\beta}\right] = \frac{1}{4}$ is the equation of the next asymptote of the cubic parallel to the basic (circle) U of the linear system of circles. But $x = \frac{1}{4}$ is the tangent at the vertex of the parabola on the origin from the center c

Hence, the system of cubics corresponding to any four tricircled lines of the cyclical parabola has the real asymptote $x = \frac{1}{4}$, the tangent at the vertex of the cyclical parabola.

The substitution of $y = mx + b$ in the Cartesian of the cubic gives

$$x^3 + x(mx+b)^2 - \alpha i(mx+b)x - \beta x^2 - \gamma(mx+b)^2 + \delta x + \epsilon(mx+b) + \rho = 0.$$

The coefficients of the two highest powers are m^2+1 and $2mb - \alpha i m - \beta - \gamma m^2$. Equating these to zero and solving, we have $m = \pm i$, $b = -\frac{1}{2}i\left[2\epsilon + \frac{1}{AB}\right]$ and $\frac{1}{2}i\left[2\bar{\epsilon} + \frac{S_4^2}{AB}\right]$.

Whence the two imaginary asymptotes are

$$y = ix - \frac{1}{2}i\left[2\epsilon + \frac{1}{AB}\right]$$

$$y = -ix + \frac{1}{2}i\left[2\bar{\epsilon} + \frac{S_4^2}{AB}\right].$$

Since δ does not involve the first term $\frac{1}{4}$ and so all cubics of the system have the same two imaginary asymptotes.

There are two quadratic systems of cir-

cles according as we change one t or two t's at a time in obtaining the orthocentric circles. These two systems are

$$(a)\quad \lambda^2 U + 2\lambda V + W = 0$$

$$(b)\quad \lambda' U' + 2\lambda' V' + W = 0$$

W, the point circle, is the common circle of the two systems, because it does not involve s_4 to the first power. Now, every circle of the two systems has its center on the corresponding parabola of centers. It must, therefore, be a point (circle) on the two center parabolas, and we have seen the tangents thereat are the two base circles (lines) of the quadratic systems, viz.,

$$s_4 x + \bar{x} - s_4 c - \bar{c} = 0$$

and

$$s_4 x - \bar{x} - s_4 c + \bar{c} = 0$$

The fixed circle which every circle of one system cuts orthogonally can be easily found. Since this fixed circle intersects orthogonally the circles (lines) U and V, it must [illegible]

centre at the intersection of these lines,
The lines $S_4 x + \bar{x} - S_4 c - \bar{c} = 0$
and $x + \bar{x} - k - c = 0$
evidently intersect at the point $\frac{S_4 c - k}{S_4 - 1}$, which is the required centre. Now, the equation of any circle is $x\bar{x} - \bar{a}x - a\bar{x} = -a\bar{a} + k\bar{k}$, and as the required circle is to cut $W \equiv c$ orthogonally, this circle is on c. Whence the fixed circle for one system is

$$x\bar{x} - \frac{S_4 c - k}{S_4 - 1}\bar{x} - \frac{\bar{k} S_4 - \bar{c}}{S_4 - 1}x = \frac{k\bar{c} - S_4 k\bar{c} + c\bar{c}(S_4 - 1)}{1 - S_4}$$

Similarly, that for the second system is

$$x\bar{x} + \frac{S_4 c + k'}{S_4 + 1}\bar{x} + \frac{\bar{k}' S_4 + \bar{c}}{S_4 + 1}x = \frac{S_4 k'\bar{c} + k'\bar{c} - c\bar{c}(1 + S_4)}{S_4 + 1}$$

The tangents at the point c of these circles are respectively

$$S_4(\bar{c} - \bar{k})x - (c - k)\bar{x} = c\bar{c}(1 - S_4) - 3(k\bar{c} - S_4 k\bar{c}),$$

$$S_4(\bar{c} - \bar{k}')x + (c - k')\bar{x} = -c\bar{c}(1 + S_4) + 3(k\bar{c} + S_4 k'\bar{c}).$$

These lines are evidently perpendicular, hence the two fixed circles, each passing through the centre of the respective

* $c - k$ is real.

quadratic systems, intersect at right angles

A particular position of this point
is, obviously, at each of the two quadratic
systems of circles that contain the
origin. The equation of the tangent is
the envelope of the system of which both
the original circle and the three others
on the moving line become infinitesimal
$$\lambda^2\bar{\kappa} + \lambda\bar{\kappa}^2 - \left[\bar{c} + \bar{c}\,(\bar{c}+K) + \frac{S_4}{AB}\right]\lambda\bar{\kappa} - \left(\bar{c} + \frac{S_4^2}{2AB}\right)\lambda^2 - \left(c + \frac{1}{2AB}\right)\bar{\kappa}^2 +$$
$$+ \left[c\,\bar{c} + \bar{c}\,(\bar{c}+K) + \frac{S_4\bar{c} + S_4\bar{c}}{AB}\right]\lambda + \left[c\,\bar{c} + c\,(\bar{c}+K) + \frac{\bar{c} + S_4 c}{AB}\right]\bar{\kappa}$$
$$- \left[c\,\bar{c}\,(K) + \frac{(\quad)}{2AB}\right]$$
; and of the attempt
to find the tangent at the point c
the resulting expression is [illegible]
c is, therefore, a singular point. The equa-
for the equation when — is the point of
[illegible] the original point [illegible]
by — by means of the relation to the point —
the equation becomes
$$\lambda^2\bar{\kappa} + \lambda\bar{\kappa}^2 + \left(\frac{-1 + \sigma_2 - \sigma_3}{2AB}\right)\lambda\bar{\kappa} - \frac{S_4^2}{2AB}\lambda^2 - \bar{\kappa}^2 -$$

it would not involving λ & μ linearly. Thus the ∞^1 quadratic systems of φ will be the same cubic envelope. It is also evident from this equation that $x_1 = x_2 = x_3$ is a double point of the cubic. We thus have the result that the envelope of the ∞^1 quadratic systems of conics is a cubical curve.

Again, the equation giving the nodal tangents, or rather the elements of the pairs, is $\left(\frac{\partial \varphi}{\partial x}\right)^2 + \lambda\left(-\frac{\partial^2 \varphi}{\partial x^2}\right)\left(\frac{\partial \varphi}{\partial x}\right) + \mu\left(\frac{\partial \varphi}{\partial x}\right)^2 = 0$. The discriminant of this quadratic does not vanish. In general, therefore, there is no nodal or cuspidal tangent. The condition for a cusp is the vanishing of this discriminant of the "first polar".

The equation of the cubic referred to any origin may be written briefly,

$$k^2 \bar{k} + k \bar{x} + \varphi \bar{x} - \bar{x}^2 = 0$$

If in this equation we write tx for $\bar{x}$, we get $tx^3 + t^2x^3 + atx^2 - bx^2 - \bar{b}t^2x^2 = 0$

or $x^2(x + t^2x + at - b - \bar{b}t^2) = 0$.

Since in the case $tx = 0$ we have $x = 0$, $x^2 = 0$, the remaining intersection is

$$x = \frac{\bar{b}t^2 - at + b}{t^2 + t},$$

while corresponding to $\bar{x}$ we have $\bar{x} = \frac{\bar{b}t^2 - at + b}{t+1}$.

These equations give a one-to-one correspondence between the points of the conic and the points of the line $Ax + B\bar{x} + 1 = 0$ intersected by the conic, the coordinates of the latter being given in terms of the parameter t:

$$A\left(\frac{\bar{b}t^2 - at + b}{t^2 + t}\right) + B\left(\frac{\bar{b}t^2 - at + b}{t+1}\right) + 1 = 0$$

or $B\bar{b}t^3 - (Ba - A\bar{b} - 1)t^2 + (B - Aa + 1)t + Ab = 0$.

Equating coefficients to the symmetric functions of the roots, we have

$$(1) \quad -A\bar{b} + (a - \bar{b}s_1)B - 1 = 0$$

$$(2) \quad -Aa + (b - \bar{b}s_2)B + 1 = 0$$

(3) $Ab + \bar{t} s_3 B = 0$

Eliminating A and B, there results

$$\begin{vmatrix} -\bar{t} & a - \bar{t} s_1 & -1 \\ -a & b - \bar{t} s_2 & 1 \\ b & \bar{t} s_3 & 0 \end{vmatrix} = 0$$

which developed is

$$(a+\bar{t})\bar{t} s_3 - b\bar{t} s_2 - b\bar{t} s_1 + (a+b) b = 0,$$

this function [illegible] permits [illegible] roots, the [illegible] roots, whence, the [illegible] giving the [illegible] in

$$t^3 - \frac{3b}{a+b} t^2 - \frac{3b}{a+b} t + \frac{(a+b) b}{(a+b)} = 0$$

Again, any cubic $Mx + Ny + 1 = 0$, M and N conjugate, will [illegible] in the three points

$$t^3 - \frac{Na - M\bar{t} - 1}{N\bar{b}} t^2 - \frac{Ma - N\bar{b} - 1}{a\bar{t}} t + \frac{M\bar{b}}{N\bar{t}} = 0$$

If this equation is to be [illegible] roots [illegible] roots, the [illegible]

[illegible]

$$\frac{M a - N b - 1}{N a - M b - 1} = \frac{3}{4} = \frac{2}{4} - 1$$

$$\frac{M a - N b - 1}{N a - M b - 1} = \frac{1}{4}$$

$$\frac{M a - N b - 1}{M a - N b - 1} = \frac{1}{2}$$

[illegible]

and $$N = \frac{a + b}{a^2 - 4bb}$$

If now these values of M and N are put in the cubic of M and N above, it reduces to the cubic giving the foci.

[illegible]

get as the equation of this locus:

$$(a+b)x + (a+b)y = 4bb - a^2,$$

Restoring the values of a and b, we have

$$\left(1 - \frac{1}{2R}\right)x + \left(1 - \frac{1}{2R}\right)y = \frac{1}{4R} - \frac{1}{4R^2},$$

the flex line referred to C as origin.

The point C has played an important part in the analysis of the en-

velope cubic and the 'focal' parabolas. We have seen that it is

The common point circle of the two quadratic systems;

One of the intersections of the focal parabolas;

The meet of the base circles (lines) $S_4 X + X_4 \epsilon - \bar{\epsilon} = 0$ and $-S_4 X + X + S_4 \epsilon - \bar{\epsilon} = 0$;

The double point of the cubic envelope;

The double points of the cubic.

The last fact is readily seen if we form the discriminant of the equation of the cubic arranged in order of powers of $\bar{x}$. The latter

$$\bar{x}^2(x - \bar{b}) + (x^2 + ax)\bar{x} - bx^2 = 0$$

whose discriminant is, equated to zero,

$$x^2\left[x^2 + 2(a+2b)x + a^2 - 4b\bar{b}\right] = 0.$$

It is evident from this equation that the origin is a double point, that is, the new cubic $\mathfrak{C}$, and not the points of the parabola in the original form Σ are

Section 4.

The Loud Circles for Four Lines of a Parabola.

The Loud circle for four lines of a parabola was: $2\chi = \frac{1}{A} + \frac{1}{B} - t\left(\frac{1}{A} - \frac{1}{B}\right)$

with center $2\chi = \frac{1}{A} + \frac{1}{B}$. Upon writing we have

$$2\chi = \frac{1-t}{A} + \frac{1+t}{B}, \quad A \text{ and } B \text{ for } S\text{'s}.$$

the Loud circle for four lines. The center is

$$2\chi = \frac{1}{A} + \frac{1}{B}$$

By reversing the lines in all ways to get real Loud circles, we get a system in all, of course, with a center, what is the distinguishing characteristic possibility one? For the purpose of answering this question, take the equation of the center $2AB\chi = B + A$

The conjugate is $-2AB\overline{\chi} = (B - A)S_S -$

$$2AB S_S - \chi = S_S(B + A)$$

$$AB(S_5 x - \bar{x}) = S_5 - B$$

$$AB(S_5 x + \bar{x}) = S_5 + A$$

$$A^2 B^2 (S_5^2 x^2 - \bar{x}^2) = S_5^2 AB$$

$$\text{or} \qquad S_5 x^2 - \bar{x} = \frac{S_5^2}{AB},$$

the curve sought. It is for a variable x a rectangular hyperbola centering at $\left(\frac{1}{S_5}\bar{x}, 0\right)$ and $\left(0, -\frac{1}{S_5}\bar{x}\right)$ if we suppose, what seems reasonable both from the form of the equation.

The asymptotes are evidently

$$x + \frac{1}{S_5}\bar{x} = 0$$

$$x - \frac{1}{S_5}\bar{x} = 0.$$

These lines, it is true, pertain to a variable, but as a pair of lines they satisfy the requirement as to change of sign if one or both S_5, instead of remaining constant change with S_5. The foci are given by

$$4 S_5^2 x^2 = \frac{4 S_5^2}{AB}$$

$$x = \pm\sqrt{\frac{1}{AB}}$$

Consider the four circle itself for the 5 lines; $2x = \frac{1}{A} + \frac{1}{B} - t\left(\frac{1}{A} - \frac{1}{B}\right)$

$$2\bar{x} = -\frac{5_5^{-}}{A} + \frac{5_5^{-}}{B} + \frac{5_5^{-}}{t}\left(\frac{1}{A} + \frac{1}{B}\right).$$

Whence, the circle in conjugate coordinates is;

$$2x\bar{x} - \left(\frac{1}{A} + \frac{1}{B}\right)\bar{x} + 5_5\left(\frac{1}{A} - \frac{1}{B}\right) = 0.$$

This circle is on the origin, and so are the fifteen other circles, while the centers, as we have just seen, are all on a rectangular hyperbola. While the circle as

$$x\bar{x} - \alpha\bar{x} - \bar{\alpha}x = 0. \quad \alpha = \frac{1}{2}\left(\frac{1}{A} + \frac{1}{B}\right)$$

The center is α and it satisfies the equation of condition

$$5_5^2 x^2 - \bar{x}^2 = \frac{5_5^2}{AB},$$

the rectangular hyperbola. Consider x and $\bar{x}$ as parameters connected by the equation just given. Then differentiating as to x, we have $\bar{x}\,d\alpha + x\,d\bar{\alpha} = 0$

$$\frac{d\bar{\alpha}}{d\alpha} = -\frac{\bar{x}}{x}$$

also from $5_5^2\alpha^2 - \bar{x}^2 = \frac{5_5^2}{AB}$ we have

$$\frac{\delta \bar{x}}{\delta x} = \frac{\bar{x}^2}{\bar{x}^2}$$

Hence $$\frac{\alpha \bar{x}^2}{\bar{x}} = -\frac{\bar{x}}{x}$$

or $$s_3^2 \alpha x + \bar{x}\, x = 0$$

Combining this with $\bar{x}\alpha + x\bar{\alpha} = x\bar{x}$ and $y = t$

$$\alpha = -\frac{x\bar{x}^2}{s_3^2 x^2 - \bar{x}^2}$$

$$\bar{\alpha} = \frac{s_3^2 x^2 \bar{x}}{s_3^2 x^2 - \bar{x}^2}$$

Substituting these values in

$$s_3^2 \alpha^2 + \bar{\alpha}^2 = s_3^2 \delta\beta$$

we have $\delta\beta\, x^2 \bar{x}^2 = \bar{x}^2 - s_3^2 x^2$,

the equation of the locus of nodes in the z-plane, — a bicircular quartic. This quartic is on the origin. The directions of the nodal tangents are given by

$$\left(\frac{dx}{d\bar{x}}\right)^2 + \frac{2(1+s_3^2)}{1-s_3^2}\left(\frac{dx}{d\bar{x}}\right) - 1$$

from which it is evident that the tangents are perpendicular. Solving for $\frac{dx}{d\bar{x}}$, we have $\frac{dx}{d\bar{x}} = \pm \frac{1}{s_3}$. Hence, the nodal tangents are

$$x = \pm \frac{1}{s_3}\bar{x},$$

which are precisely the asymptotes of

the rectangular hyperbola. If these two points be combined with the equation of the points we get

$$x^4 = c$$

which shows that this locus touches the axes at the four points. Hence, the inverse is a lifecanodrome with its crunodes at the points of the axes.

Thus, the envelope of the system of circles to which the sixteen Fourdee circles for 5 lines of a parabola belong is the lemniscate.

Again, the system to which belong the Fourdee circles above is a quadratic system.

It may be added here that six lines of a parabola give results quite similar those of 4 lines. For instance, the centers of Fourdee circles for any seven

lines lie on the perpendicular lines.

Section 5.

The Locus Circles for Five Lines of a Double Parabola.

Since a double parabola is a class cubic, and since it touches the line at infinity as often as possible, that is to say twice, which counts for three conditions, there remain $\frac{1}{2} \cdot 3 \cdot (3+3) - 3$ conditions, or 6 conditions to be satisfied. Hence 6 lines uniquely determine a double parabola, and we may, if we choose, use the line equation of this curve in any investigation for metrical properties of a 5-line or 6-line, the lines being described.

Consider first the formation of the pencil of the double parabola. Take the

equation

$$K = \frac{c_1}{(\lambda_1 - t)^2} + \frac{c_2}{(\lambda_2 - t)^2}$$

in which c_1 and c_2 are arbitrary complex numbers, and λ_1 and λ_2 are arbitrary curves. This equation is not yet that of a double-point curve, but must have the conditions for cusps put upon it. The condition must be

$$\frac{c_1'}{(\lambda_1' - t)^2} + \frac{c_2'}{(\lambda_2' - t)^2} \equiv \frac{-\bar{c}_1 \lambda_1'^2 t'}{(\lambda_1' - t)^2} + \frac{-\bar{c}_2 \lambda_2'^2 t'}{(\lambda_2' - t)^2}$$

that is, $\bar{c}_i = \frac{c_i}{\lambda_i'}$

The first equation of the tangent at t_1 is

$$x = \frac{c_1'}{(\lambda_1' - t_1)^2 (\lambda_1' - \bar{t}_1)} + \frac{c_2'}{(\lambda_2' - t_1)(\lambda_2' - \bar{t}_1)}$$

which, when t varies, is a hyperbolic locus of a certain degree. It apparently goes off to infinity in two directions, but along one locus the same. From the equation of the parabola it is seen that for $t = \lambda_1$, and $t = \lambda_2$, it becomes infinite. What is true

$$x = \frac{c_1}{(\lambda_1 - t_1)(\lambda_1 - t)} + \frac{c_2}{(\lambda_2 - t_1)(\lambda_2 - t)}$$

and $$\bar{x} = \frac{c_1 t_1 t}{t_1(\lambda_1 - t_1)(\lambda_1 - t)} + \frac{c_2 t_1 t}{\lambda_2(\lambda_2 - t_1)(\lambda_2 - t)},$$

dividing x by $\bar{x}$ and taking the square root of the quotient, we have either

$$\sqrt{\frac{\lambda_1 \lambda_2}{t_1 t} \times \frac{c_1 + \frac{c_2(\lambda_1 - t_1)(\lambda_1 - t)}{(\lambda_2 - t_1)(\lambda_2 - t)}}{c_1 \lambda_1 + \frac{c_2 \lambda_1 (\lambda_1 - t_1)}{(\lambda_2 - t_1)(\lambda_2 - t)}}}$$

or

$$\sqrt{\frac{\lambda_1 \lambda_2}{t_1 t} \times \frac{\frac{c_1(\lambda_2 - t_1)(\lambda_2 - t)}{(\lambda_2 - t_1)(\lambda_1 - t)} + c_2}{\frac{c_1 \lambda_1 (\lambda_2 - t_1)(\lambda_2 - t)}{(\lambda_2 - t_1)(\lambda_1 - t)} + c_2 \lambda_1}}.$$

If we put $t = t_1$, or $t = \lambda_2$ be substituted, in either expression respectively, we have in each case $\sqrt{\frac{1}{t_1}}$. Hence the two directions are opposites and the original equation is that of a circle.

Next combine the equations of x and $\bar{x}$, we have then

$$c_1 x - \bar{x} = \frac{c_1 t_1}{\lambda_1(\lambda_1 - t_1)} + \frac{c_2 t_1}{\lambda_2(\lambda_2 - t_1)},$$

this equation by clearing of fractions and [illegible] for which is $x = \bar{x}$ [illegible]. If we put for t_1 by t_1' and λ_2 by λ_2' [illegible]

(symmetry and reference), we have

$$t_1 x + \frac{\bar{x}}{t_1} = \frac{c_1 t_1}{\lambda_1^2(\lambda_1^2 - t_1^2)} + \frac{c_2 t_1}{\lambda_2^2(\lambda_2^2 - t_1^2)},$$

the director line of the curve for the parameter t_1. The directrix of the curve is

$$x = \frac{c_1}{\lambda_1^2 - t^2} + \frac{c_2}{\lambda_2^2 - t^2}$$

The next work is to obtain the equidistant [illegible] of three lines of the curve. The equidistant axis for lines t_1 and t_2 is

$$t_1 t_2 (t_1 - t_2) x + (t_1 - t_2)\bar{x} = \sum^2 \frac{\left[c_1 t_1(\lambda_1^2 - t_2^2) - c_1 t_2(\lambda_1^2 - t_1^2)\right] t_1 t_2}{\lambda_1^2(\lambda_1^2 - t_1^2)(\lambda_1^2 - t_2^2)}$$

$$\text{or} \quad t_1 t_2 x + \bar{x} = \sum^2 \frac{c_1(\lambda_1^2 + t_1 t_2) t_1 t_2}{\lambda_1^2(\lambda_1^2 - t_1^2)(\lambda_1^2 - t_2^2)}$$

The equidistant axis for lines t_1 and t_3 is

$$t_1 t_3 x + \bar{x} = \sum^2 \frac{c_1 (\lambda_1^2 + t_1 t_3) t_1 t_3}{\lambda_1^2(\lambda_1^2 - t_1^2)(\lambda_1^2 - t_3^2)}.$$

Combining these two equations and dividing by $t_2 - t_3$, the result is

$$(t_2 - t_3) x = \sum^2 \frac{c_1 \left[(\lambda_1^2 + t_2 t_3)(\lambda_1^2 - t_1^2) - (\lambda_1^2 + t_1 t_3)(\lambda_1^2 - t_2^2)\right] t_1}{\lambda_1^2(\lambda_1^2 - t_1^2)(\lambda_1^2 - t_2^2)(\lambda_1^2 - t_3^2)}$$

$$x = \sum^2 \frac{c_1 \lambda_1^2 (\lambda_1^2 + t_1 t_2 + t_1 t_3 + t_2 t_3)}{\lambda_1^2(\lambda_1^2 - t_1^2)(\lambda_1^2 - t_2^2)(\lambda_1^2 - t_3^2)}$$

or $$x = \sum^{2} \frac{c_1(\lambda_1^2 + s_2)}{(\lambda_1^2 - t_1^2)(\lambda_1^2 - t_2^2)(\lambda_1^2 - t_3^2)}$$

$$(\lambda_1^2 - t_1^2)(\lambda_1^2 - t_2^2)(\lambda_1^2 - t_3^2) = [(\lambda_1 - t_1)(\lambda_1 - t_2)\lambda_1 - t_3][(\lambda_1 + t_1)(\lambda_1 + t_2)(\lambda_1 + t_3)]$$

$$= A_1 B_1, \text{ say.}$$

Then $$\frac{1}{A_1} + \frac{1}{B_1} = \frac{2\lambda_1(\lambda_1^2 + s_2)}{A_1 B_1}$$

Likewise $$\frac{1}{A_2} + \frac{1}{B_2} = \frac{2\lambda_2(\lambda_2^2 + s_2)}{A_2 B_2}$$

Therefore $$x = \frac{c_1}{2\lambda_1}\left(\frac{1}{A_1} + \frac{1}{B_1}\right) + \frac{c_2}{2\lambda_2}\left(\frac{1}{A_2} + \frac{1}{B_2}\right)$$

or $$2x = \sum^{2} \frac{c_1}{\lambda_1}\left(\frac{1}{A_1} + \frac{1}{B_1}\right),$$

the equidistant pencil of the three lines t_1, t_2, t_3. The pencil for Ξ of 4 lines is

$$2x = \sum^{2} \lambda_1 \left(\frac{\lambda_1 - t}{A_1} + \frac{\lambda_1 + t}{B_1}\right), \quad A = R(\lambda_1 + t'), \quad B = R(\lambda_1 - t')$$

Thus is the form of circle for four lines of the double pencil, with center

$$2\bar{x} = \sum^{2} c_1\left(\frac{1}{A_1} + \frac{1}{B_1}\right).$$

The circle for five lines, then, is

$$2x = \sum^{2} \frac{c_1}{\lambda_1}\left(\frac{\lambda_1 - t}{A_1} + \frac{\lambda_1 + t}{B_1}\right), \quad A = R(\lambda_1 - t'), \quad B = R(\lambda_1 + t')$$

with center

$$2\bar{x} = c_1\left(\frac{1}{A_1} + \frac{1}{B_1}\right) + c_2\left(\frac{1}{A_2} + \frac{1}{B_2}\right)$$

We have already seen that by inversion

the lines, we get fifteen other similar expressions for cevians. What is the simplest curve these sixteen cevians are on? We have answered this question when the first lines were lines of a single parabola, and therefore subject to a condition; but as the first lines now are lines of double parabola, we are using the lines without any condition on them. To answer the question above we must obtain an expression for cevian involving the even powers of t_2 or of symmetric functions of the same.

The expression for cevian may be written

$$\Sigma\chi = c_1\left(\frac{\lambda_1^5 + \lambda_1^3 s_2 + \lambda_1 s_4}{A_1 B_1}\right) + c_2\left(\frac{\lambda_2^5 + \lambda_2^3 s_2 + \lambda_2 s_4}{A_2 B_2}\right)$$

$$\text{or } \chi - \left(\frac{c_1\lambda_1^5}{A_1B_1} + \frac{c_2\lambda_2^5}{A_2B_2}\right) = \left(\frac{c_1\lambda_1^3}{A_1B_1} + \frac{c_2\lambda_2^3}{A_2B_2}\right)s_2 + \left(\frac{c_1\lambda_1}{A_1B_1} + \frac{c_2\lambda_2}{A_2B_2}\right)s_4$$

$$\text{or } \frac{\chi - \left(\frac{c_1\lambda_1^5}{A_1B_1} + \frac{c_2\lambda_2^5}{A_2B_2}\right)}{\left(\frac{c_1\lambda_1}{A_1B_1} + \frac{c_2\lambda_2}{A_2B_2}\right)} = \frac{\left(\frac{c_1\lambda_1^3}{A_1B_1} + \frac{c_2\lambda_2^3}{A_2B_2}\right)}{\left(\frac{c_1\lambda_1}{A_1B_1} + \frac{c_2\lambda_2}{A_2B_2}\right)}s_2 + s_4$$

The conjugate of the coefficient of S_2 is

$$\frac{\frac{c_1\lambda_1}{A_1B_1} + \frac{c_2\lambda_2}{A_2B_2}}{\frac{c_1\lambda_1^3}{A_1B_1} + \frac{c_2\lambda_2^3}{A_1B_2}},$$

that is, the coefficient of S_2 is a turn. Call it $\frac{\xi}{m} \equiv \mu^2$. Write for shortness the above equation in the form

$$\frac{X-K}{m} = \mu^2 S_2 + S_4$$

Now, $(\mu^5 + \mu^4 S_1 + \mu^3 S_2 + \mu^2 S_3 + \mu S_4 + S_5) + (\mu^5 - \mu^4 S_1 + \mu^3 S_2 - \mu^2 S_3 + \mu S_4 - S_5)$

$= 2\mu^5 + 2\mu^3 S_2 + 2\mu S_4$. Hence $\mu^2 S_2 + S_4$ is equal to $\frac{\overset{5}{\Pi}(\mu+t) + \overset{5}{\Pi}(\mu-t)}{2\mu} - \mu^4$,

and $\frac{X-K}{m} + \mu^4 = \frac{\overset{5}{\Pi}(\mu+t) + \overset{5}{\Pi}(\mu-t)}{2\mu}$

or $\frac{2\mu(X-K)}{m} + 2\mu^5 = \overset{5}{\Pi}(\mu+t) + \overset{5}{\Pi}(\mu-t)$

The conjugate of $\mathcal{M} \equiv \frac{c_1\lambda_1}{A_1B_1} + \frac{c_2\lambda_2}{A_2B_2}$ is

$-S_5^2\left(\frac{c_1\lambda_1^3}{A_1B_1} + \frac{c_2\lambda_2^3}{A_2B_2}\right) \equiv -S_5^2\mathcal{L}$. Hence taking

a conjugate of the equation, we have

$$\frac{2(\overline{X}-\overline{K})}{-\mu\mathcal{L}} + \frac{2S_5^2}{\mu^5} = \frac{S_5\overset{5}{\Pi}(\mu+t) - S_5\overset{5}{\Pi}(\mu-t)}{\mu^5}$$

or $\frac{2\mu^4(\overline{X}-\overline{K})}{-\mathcal{L}} + 2S_5^2 = S_5\overset{5}{\Pi}(\mu+t) - S_5\overset{5}{\Pi}(\mu-t)$

Adding and subtracting this equation and its conjugate [illegible]

following equations;

(a) $\frac{\mu s_2(\bar{x}-\bar{h})}{m} + \mu^5 s_3 - + \frac{\mu^4(\bar{x}-\bar{k})}{-\mathcal{L}} + s_3^2 = s_3 \dot{\pi}(\mu+t)$

(b) $\frac{\mu s_2(\bar{x}-\bar{k})}{m} + \mu^5 s_3 - \frac{\mu^4(\bar{x}-\bar{k})}{-\mathcal{L}} + s_3^2 = s_3 \dot{\pi}(\mu - t)$

Multiplying these equations, (a) and (b), together member by member, there results

$$\mu^2 s_3^2 \left(\frac{\bar{x}-\bar{k}}{m} + \mu^4\right)^2 - \left(\frac{(\bar{x}-\bar{k})\mu^4}{-\mathcal{L}} + s_3^2\right)^2 = s_3^2 \dot{\pi}(\mu^2 - t^2),$$

an equation unaffected by any change in sign of the t's, the parameters of the first term. But it is by changing the sign of the parameter that we reverse the direction of the curve. Therefore this equation (c) for a variable $\bar{x}$ represents the entire system of curves under consideration.

Expanding (c) we have,

$$\mu^2 s_3^2 \mathcal{L}^2 \bar{x}^2 - \mu^8 m^2 \bar{x}^2 + (2\mu^6 s_3^2 m \mathcal{L}^2 - 2\mu^4 s_3^2 \mathcal{L}^2 \bar{k})\bar{x}$$
$$+ (2\mu^4 s_3^2 \mathcal{L} m^2 + \mu^8 m^2 \bar{k}) \bar{x} = s_3 \mathcal{L}^2 m^2 \mathcal{G} - \mu^2 s_3^2 \mathcal{L}^2 \bar{k}^2$$

$+2\mu^2 S_5^2 \lambda \mathcal{L} l m - \mu^2 S_5^2 \lambda^2 m^2 + l^2 \lambda^2 m^2 + 2\mu^4 S_5^2 \lambda \mathcal{L} m^2$
$+ S_5^4 \mathcal{L}^2 m^2$, where $\mathcal{Q} = S_5 - \pi(\mu^2 - \lambda^2)$,
briefly written in conjugate coordinates
the equation is

$$a x^2 - b y^2 + c x + d y - f = 0,$$

and in Cartesians it becomes

$$x^2 + 2\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} x y - y^2 + \frac{(c+d)}{a-b} x + \frac{(c-d)}{a-b} i y - \frac{f}{a-b} = 0$$

which every coefficient is real. This is evidently the equation of a rectangular hyperbola. Hence, the centres of the Sixteen Focal circles for four lenses are on a rectangular hyperbola.

Without giving much of this work in detail it might be of interest to give the equations of the orthocentric circles for 4 and 6 lines of the double pentahedron and the directrix for 5 lines.

The line equation of the orthocentric circle for 4 lines is

$$2t^2\xi + 2\bar{x} = \sum \frac{c_1[\lambda_1^2 t^2 - \lambda_1 s_1 t^2 + \lambda_1 s_2 t - \lambda_1 s_3 + s_4]}{\lambda_1^2 A_1} + \frac{c_1[\lambda_1 t^2 + \lambda_1 s_1 t - \lambda_1 s_2 t + \lambda s_3 + s_4}{\lambda_1^2 B_1}.$$

The point equation is

$$2x = \sum \frac{c_1(\lambda_1 - s_1)}{\lambda_1 A_1} + \frac{c_1(\lambda_1 + s_1)}{\lambda_1 B_1} + \frac{c_1 s_2}{2t\lambda_1}\left(\frac{1}{A_1} - \frac{1}{B_1}\right).$$

The line equation of the orthocentric circle for 6 lines is

$$t^2\left[x + c_1\lambda_1^2 k_1 - c_1\lambda_1 s_1 k_1 + c_1 s_2 k_1 + c_2\lambda_2^2 k_2 - c_2\lambda_2 s_1 k_2 + c_2 s_2 k_2\right] + \left[\bar{x} - \frac{c_1 s_2}{\lambda_1}\xi_1 + \frac{c_1 s_6}{\lambda_1^2} k_1 + c_1 s_4 k_1 - \frac{c_2 s_2}{\lambda_2}\xi_2 + \frac{c_2 s_6}{\lambda_2^2} k_2 + c_2 s_4 k_2\right] = \left[c_1 s_3 k_1 + c_2 s_3 k_2\right] t$$

The point equation is

$$x = \sum c_1\lambda_1 s_1 \xi_1 - c_1\lambda_1^2 k_1 - \xi_1\lambda_1 k_1 + \frac{c_1 s_3 k_1}{t},$$

where in the last two equations $k_1 = \frac{1}{\lambda_1}$.

and $\lambda_i = \frac{1}{A_i} - \frac{1}{B_i}$.

If in the line equation of the orthocentric circle for four lines we call the parameter l and symmetric for four lines, we get, after some reduction, the following equation;

$$2T^2 k + \bar{k} = \frac{c_1(\lambda_1 T^2 + \lambda_1 S_1 t^2 - \lambda_1^2 S_1 T - \lambda_1 S_2 T + \lambda_1^2 S_2) t}{\lambda_1^2 A_1}$$

$$+ \frac{c_1(-\lambda_1^2 T^2 + \lambda_1 S_1 T^2 + \lambda_1^2 S_1 T - \lambda_1 S_2 T - \lambda_1 S_2) t}{\lambda_1^2 B_1}$$

$$+ \frac{c_1(-\lambda_1 T^2 + \lambda_1^2 T + 2\lambda_1 S_1 T - \lambda_1^2 S_1) t^2}{\lambda_1^2 A_1} + \frac{c_1(-\lambda_1 T^2 - \lambda_1^2 T + \lambda_1 S_1 T + \lambda_1^2 S_1) t^2}{\lambda_1^2 B_1}$$

$$+ \frac{c_2(-\lambda_2 T^2 + \lambda_2^2 T + \lambda_2 S_1 T - \lambda_2^2 S_1) t^2}{\lambda_2^2 A_2} + \frac{c_2(-\lambda_2 T^2 - \lambda_2^2 T + \lambda_2 S_1 T + \lambda_2^2 S_1) t^2}{\lambda_2^2 B_2}$$

$$+ \frac{c_1(-\lambda_1 T + \lambda_1^2) t^3}{\lambda_1^2 A_1} + \frac{c_1(-\lambda_1 T - \lambda_1^2) t^3}{\lambda_1^2 B_1} + \frac{c_2(-\lambda_2 T + \lambda_2^2) t^3}{\lambda_2^2 A_2} + \frac{c_2(-\lambda_2 T - \lambda_2^2) t^3}{\lambda_2^2 B_2}$$

$$+ \frac{c_1(\lambda_1^3 T^2 - \lambda_1^3 S_1 T^2 + \lambda_1^2 S_2 T - \lambda_1^2 S_3 + \lambda_1 S_4 - S_5)}{\lambda_1^2 A_1} + \frac{c_1(\lambda_1^3 T^2 + \lambda_1^3 S_1 T^2 - \lambda_1^2 S_2 T + \lambda_1^2 \cdot + \lambda_1 S_4 + S_5)}{\lambda_1^2 B_1}$$

$$+ c_2\left(\frac{\lambda_2^3 T^2 - \lambda_2^3 S_1 T^2 + \lambda_2 S_2 T - \lambda_2^2 S_3 + \lambda_2 S_4 - S_5)}{\lambda_2^2 A_2} + \frac{c_2(\lambda_2^3 T^2 + \lambda_2^3 S_1 T^2 - \lambda_2^2 S_2 T + \lambda_2^2 S_3 + \lambda_2 S_4 + \ldots}{\lambda_2^2 B_2}\right.$$

If we let $\left(\frac{c_i}{\lambda_i A_i} + \frac{c_i}{\lambda_i B_i}\right) \equiv m_i$ and $\left(\frac{c_i}{A_i} - \frac{c_i}{B_i}\right) \equiv n_i$

the above equation can be written

$$2T^2 k + 2\bar{k} = c_1 \lambda_1 T^2 \left(\frac{1}{A_1} + \frac{1}{B_1}\right) + c_2 \lambda_2 T^2 \left(\frac{1}{A_2} + \frac{1}{B_2}\right) - S_1 T^2 (m_1 + m_2) + S_1 T (n_1 + n_2)$$

$$- S_3 (n_1 + n_2) + S_4 (m_1 + m_2) - \frac{S_5 n_1}{\lambda_1^2} - \frac{S_5 n_2}{\lambda_2^2}.$$

$$+ \left[-S_1 T^2 (m_1 + m_2) + S_1 (n_1 + n_2) - S_2 T (m_1 + m_2) + S_2 (n_1 + n_2) \right] t$$

$$+\left[-T^2(m_1+m_2)+T(n_1+n_2)+\dots\right]t$$

$$+\left[-T(m_1+m_2)+(n_1+n_2)\right]t^3.$$

If $m_1+m_2=0$, then $T=\frac{n_1+n_2}{\dots}$, the [illegible] formula [illegible], and the equation finally becomes

$$2\left(\frac{n_1+n_2}{m_1+m_2}\right)^2 x+\dots=\left[\lambda_1\left(\frac{1}{A_1}+\frac{1}{B_1}\right)+\lambda_2\left(\frac{1}{A_2}+\frac{1}{B_2}\right)\right]\left(\frac{n_1+n_2}{m_1+m_2}\right)^3$$

$$-\frac{4(n_1+n_2)^3}{(m_1+m_2)^2}+\frac{2(n_1+n_2)^2}{(m_1+m_2)}-\beta_3(n_1+n_2)+\beta_4(m_1+m_2)$$

$$-\frac{\lambda_2^2}{\lambda_1^2}n_1-\frac{\lambda_2^2}{\lambda_2^2}n_2,$$

the directrix for T lines of the third parabola.

Section 6.

The Second Order for Six Arbitrary Lines.

Since a double parabola is uniquely determined by six lines, one of which can be any line and the other five [illegible]

necked by many relations. [illegible]

The centre of the bond circle for 6 lines when line 1, say, is reversed is

$$2x = c_1\left(\frac{1}{A_1} + \frac{1}{B_1}\right) + c_1\left(\frac{1}{A_2} + \frac{1}{B_2}\right)$$

The conjugate of this equation is

$$2\bar{x} = -S_6\left[c_1\left(\frac{1}{A_1} + \frac{1}{E}\right) + c_2\left(\frac{1}{A_2} + \frac{1}{B_2}\right)\right]$$

Hence we obtain immediately the fact that the centre of this bond junction circles have their centres on the line $S_6 x + \bar{x} = 0$

The centre of the original circle was on the line $S_6 x - \bar{x} = 2$.

These lines are perpendicular.

The centre of the circle for a reversal of lines 1 and 2 is

$$2x = c_1\left(\frac{1}{A_1} + \frac{1}{B_1}\right) + c_2\left(\frac{1}{A_2} + \frac{1}{B_2}\right)$$

$$2\bar{x} = S_6\left[c_1\left(\frac{1}{A_1} + \frac{1}{E_1}\right) + c_2\left(\frac{1}{A_2} + \frac{1}{E_2}\right)\right]$$

It is evident that the [illegible] for a reversal of two lines will [illegible] in the limit $\delta_2 K = \overline{K}$ —, the limit for the number of [illegible].

The curves for a reversal of 3 lines, say [illegible],

$$\delta K = \delta_1 \left(\frac{1}{\overline{B_1}} - \frac{1}{\overline{A_1}} \right) + \delta_2 \left(\frac{1}{\overline{B_2}} - \frac{1}{\overline{A_2}} \right)$$

$$\delta \overline{K} = \delta_1 \left(\frac{1}{B_1} - \frac{1}{A_1} \right) + \delta_2 \left(\frac{1}{B_2} - \frac{1}{A_2} \right)$$

It is plain that the 10 curves for a [illegible] on the line $\delta_6 K + \overline{K} = 0$, [illegible] for the 3rd curve = for a change [illegible]. Hence [illegible] any three [illegible] for any two [illegible] the line [illegible], to the [illegible].

Johns Hopkins University, Baltimore, Md.,
March, 1910.

Biographical Note.

Joseph Ellis Rodgers was born Jan. 4, 1872 at Chambersville, Virginia. He received his early education in the public schools of Frederick County near Winchester, Virginia. After teaching several years in the public schools, he entered Washington and Lee University from which he was graduated in 1898, having meanwhile served three years as principal of the High School of Lexington, Va. He was successively principal of the Potomac Academy, Romney, W. Va., instructor in mathematics at the Keyser Branch of the West Virginia University, and also at the Davis and Elkins College, Elkins, W. Va. He spent one year of absence the year 1903-4 in the Department of Mathematics of the Johns Hopkins University, and received graduate studies there during the sessions of 1906-7, 1907-8. During the past two years he has been instructor in mathematics at the Baltimore Polytechnic Institute, [illegible]